

Scandinavian Logic Symposium (SLSS 2022)
(Abstracts booklet) ¹

17th - 19th June, 2022. Bergen, Norway

¹*Date: 18th June 2022, 12:52.*

Contents

Contents	1
1 Invited talks	3
Explanatory proofs: philosophical framework, core ideas and results <i>Francesca Poggiolesi</i>	3
A family of modal fixpoint logics <i>Yde Venema</i>	3
Constructive mathematics from atoms to universes: In memoriam Erik Palmgren <i>Peter LeFanu Lumsdaine</i>	4
2 Contributed talks	5
A terminating intuitionistic calculus <i>Giulio Fellin, Sara Negri</i>	6
Finite Belief Base Change via Models <i>Ricardo Guimarães, Ana Ozaki, Jandson S. Ribeiro</i>	8
Introducing Proof Tree Automata and Proof Tree Graphs <i>Valentin D. Richard</i>	10
On extracting variable Herbrand disjunctions <i>Andrei Sipoş</i>	13
On Various Weak First-Order Theories <i>Lars Kristiansen</i>	15
Logic of Sentential Operators <i>Michał Walicki</i>	17
Weak Essentially Incomplete Theories of Concatenation <i>Juvenal Murwanashyaka</i>	19
Generalizing Epistemic Updates <i>Igor Sedlár</i>	21
Purity of Formal Proofs <i>Robin Martinot</i>	24
Carnap’s Problem for Generalised Quantifiers <i>Sebastian G.W. Speitel</i>	26
Extracting Rules from Neural Networks with Partial Interpretations <i>Cosimo Persia, Ana Ozaki</i>	28

Almost negative truth and fixpoints in intuitionistic logic <i>Mattias Granberg Olsson, Graham Leigh</i>	32
Approximating trees as coloured linear orders and complete axiomatisations of some classes of trees <i>Valentin Goranko, Ruaan Kellerman</i>	33
Proof-theoretic Semantics in Sheaves (Extended Abstract) <i>David Pym, Eike Ritter, Edmund Robinson</i>	35
Monotonicity under Knowledge <i>Jialiang Yan, Maria Aloni</i>	38
Generalizing Rules via Algebraic Constraints (Extended Abstract) <i>Alexander V. Gheorghiu, David J. Pym</i>	41
'New Foundations' for Metaphysics <i>Violeta Conde, Alejandro Gracia-Di Rienzo</i>	44
An Ouroboros of Team Temporal Logics <i>Max Sandström</i>	48
Modal team logics for modelling free choice inference <i>Aleksi Anttila, Maria Aloni, Fan Yang</i>	50
Demystifying Attestation in Intel Trust Domain Extensions (TDX) via Formal Verification <i>Muhammad Usama Sardar, Christof Fetzer</i>	51
Proof theoretic relations between different versions of Higman's and Kruskal's theorem <i>Gabriele Buriola, Peter Schuster</i>	53
Geometric Models and Cone Semantics: An Overview <i>Victor Lacerda, Ricardo Guimarães, Ana Ozaki</i>	55
Product of effect algebras with $\sqrt{}$ and their Logic <i>Konstantin Shishov</i>	57
A Librationist Set Theory Extending Classical Set Theory <i>Frode Alfson Bjørdal</i>	59

Chapter 1

Invited talks

Explanatory proofs: philosophical framework, core ideas and results

Francesca Poggiolesi

Since Aristotle, mathematicians and philosophers distinguish between two kinds of mathematical proofs: proofs that show a theorem to be true, and proofs that explain why the theorem is true. Whilst the former have been intensively studied and successfully formalized, e.g. via the notion of derivability in natural deduction calculi, the latter have been mainly ignored or forgotten. In this talk we aim at moving the first steps towards repairing this situation. After a philosophical introduction to explanatory proofs, we provide what we believe is a good characterization and formalization of them.

We conclude the talk by offering a plethora of results/directions of future research that explanatory proofs lead to.

A family of modal fixpoint logics

Yde Venema

Modal fixpoint logics are extensions of basic modal logic, either with fixpoint connectives, such as the common knowledge operator in epistemic logic or the until operator in temporal logic, or with explicit least- and greatest fixpoint operators, as in the modal mu-calculus. Such formalisms significantly increase the expressive power of the language by enabling the expression of recursive phenomena.

In the talk I will discuss a small family of modal fixpoint logics that we obtain by syntactically restricting the application of the fixpoint operators in the modal mu-calculus. This family contains some interesting and well-known members, such as propositional dynamic logic and the alternation-free mu-calculus. We will review some recent results on the model theory and the proof theory of this family,

In the talk I will assume some rudimentary knowledge of basic modal logic, but no prior acquaintance with the modal mu-calculus.

**Constructive mathematics from atoms to universes: In memoriam
Erik Palmgren**

Peter LeFanu Lumsdaine

Our colleague Erik Palmgren died unexpectedly in November 2019 – just before Covid closed down conferences. In this belated memorial lecture, I will survey Erik’s work, which ranged over type theory, constructive mathematics, proof theory, and categorical logic. In particular I will try to show something of his characteristic mathematical worldview, as I heard it from him while we were colleagues in Stockholm.

Chapter 2

Contributed talks

A terminating intuitionistic calculus

Giulio Fellin^{*†‡} and Sara Negri[§]

In his doctoral thesis [6,7], Gentzen introduced sequent calculi for classical and intuitionistic logic. In particular, he solved the decision problem for intuitionistic propositional logic **Int** with a calculus that he called **LI**. However, Gentzen's original calculus lacked some desirable properties, such as the invertibility of rules which would eliminate the need for backtracking. Ever since then, many other approaches were proposed; we refer to [3] for an extended survey.

The labelled calculus **G3I** by Dyckhoff and Negri [4,8,11] solves the problem of backtracking but loses the property of termination, for example in the case of Peirce's Law. In order to solve this problem, Negri [9,10] showed how to add a loop-checking mechanism to ensure termination. However, it is desirable to avoid loop-checking since its effect on complexity isn't clear.

Corsi [1,2] presented a calculus for **Int** which fulfils the termination property. The key to get termination is the addition of the following rule:

$$\frac{\Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \supset B} \textit{ a fortiori}$$

This rule is logically equivalent to the formula $B \supset (A \supset B)$, which is the principle of *a fortiori*.

In the present paper, we consider the labelled calculus **G3I** instead, and show that, a way to reach termination consists in modifying rule $R\supset$ as follows:

$$\frac{x \leq y, y: B \supset (A \supset B), y: A, \Gamma \rightarrow \Delta, y: B}{\Gamma \rightarrow \Delta, x: A \supset B} R\supset_t \quad (y \text{ fresh})$$

We call the resulting calculus **G3I_t**.

We show that, given a sequent $\Gamma \rightarrow \Delta$ in the language of **G3I_t**, it is decidable whether it is derivable in **G3I_t**. This is done by means of a proof search algorithm which is based on the one that Dyckhoff and Negri [5] gave for the calculus **G3Grz** for the provability logic **Grz**, into which there **Int** is embeddable. Moreover, if $\Gamma \rightarrow \Delta$ is not derivable, then the failed proof search gives a finite countermodel to the sequent on a reflexive, transitive and Noetherian Kripke frame. Although the idea comes from **G3Grz**, we notice that what we actually do is incorporating *a fortiori* into $R\supset$.

References

- [1] Giovanna Corsi. The *a fortiori* rule: the key to reach termination in intuitionistic logic. In E. Ballo and M. Franchella, editors, *Logic and Philosophy in Italy. Some Trends and*

^{*}Università di Verona, Dipartimento di Informatica, Strada le Grazie 15, 37134, Verona, Italy. E-mail: giulio.fellin@univr.it

[†]Università di Trento, Dipartimento di Matematica, Trento, Italy.

[‡]University of Helsinki, Department of Philosophy, Helsinki, Finland.

[§]Università di Genova, Dipartimento di Matematica, via Dodecaneso, 35, 16146, Genova, Italy. E-mail: sara.negri@unige.it

- Perspectives. Essays in Honor of Corrado Mangione on his 75th Birthday*, pages 27–47. Polimettrica, Milano, 2006.
- [2] Giovanna Corsi and Gabriele Tassi. Intuitionistic logic freed of all metarules. *Journal of Symbolic Logic*, 72(4):1204–1218, 2007. doi:10.2178/js1/1203350782.
 - [3] Roy Dyckhoff. Intuitionistic decision procedures since gentzen. In R. Kahle, T. Strahm, and T. Studer, editors, *Advances in Proof Theory. Progress in Computer Science and Applied Logic*, volume 28. Birkhäuser, Cham, 2016. doi:978-3-319-29198-7_6.
 - [4] Roy Dyckhoff and Sara Negri. Proof analysis in intermediate propositional logics. *Archive Math. Logic*, 51(1–2):71–92, 2012.
 - [5] Roy Dyckhoff and Sara Negri. A cut-free sequent system for Grzegorzcyk logic, with an application to the Gödel–McKinsey–Tarski embedding. *J. Logic Comput.*, 26(1):169–187, 2016.
 - [6] Gerhard Gentzen. Untersuchungen über das logische Schließen I. *Math. Z.*, 39:176–210, 1934.
 - [7] Gerhard Gentzen. Untersuchungen über das logische Schließen II. *Math. Z.*, 39:405–431, 1934.
 - [8] Sara Negri. Proof analysis in non-classical logics. In *Logic Colloquium 2005*, volume 28 of *Lect. Notes Log.*, pages 107–128. Assoc. Symbol. Logic, Urbana, IL, 2008.
 - [9] Sara Negri. On the duality of proofs and countermodels in labelled sequent calculi. In *TABLEAUX*, 2013.
 - [10] Sara Negri. Proofs and countermodels in non-classical logics. *Log. Univers.*, 8(1):25–60, 2014.
 - [11] Sara Negri and Jan von Plato. *Proof Analysis. A Contribution to Hilbert’s Last Problem*. Cambridge University Press, Cambridge, 2011. doi:10.1017/CB09781139003513.

Finite Belief Base Change via Models

Ricardo Guimarães¹, Ana Ozaki¹, and Jandson S. Ribeiro²

¹University of Bergen

²FernUniversität Hagen

This extended abstract is a summary of a paper under review. Belief Change [Alchourrón et al., 1985, Hansson, 1999] is the study of how a rational agent should autonomously modify its current beliefs in response to new information. In all cases, the agent should minimally change beliefs that conflict with the new information. The principle of minimal change is captured in Belief Change via rationality postulates that dictate the properties of a rational change. The main paradigms of Belief Change assume that the agent’s beliefs are represented as a *set of formulae* expressed in some underlying logic, such as propositional logics; while incoming information is represented as formulae in the same logic.

In this work, we introduce and investigate a new paradigm for Belief Change where incoming pieces of information are (possibly infinite) *models*, while the agent’s body of knowledge is represented as a *finite* set of formulae. We impose the finite representability requirement for the sets of formulae because reasoners for a particular logic usually can only deal with finite sets of formulae. Here, the power set of a set A is denoted by $\mathcal{P}(A)$, while all finite subsets of A is given by $\mathcal{P}_f(A)$.

Following Aiguier et al. [2018] and Delgrande et al. [2018], we look at a logic as a satisfaction system. A satisfaction system is a triple $\Lambda = (\mathcal{L}, \mathfrak{M}, \models)$, where \mathcal{L} is a language, \mathfrak{M} is the set of all possible models, also called interpretations, used to give meaning to the sentences in \mathcal{L} , and \models is a satisfaction relation which indicates that a model M satisfies a base \mathcal{B} (in symbols, $M \models \mathcal{B}$). Also, given a satisfaction system $\Lambda = (\mathcal{L}, \mathfrak{M}, \models)$, we define $\text{Mod}(\mathcal{B}) := \{M \in \mathfrak{M} \mid M \models \mathcal{B}\}$. Additionally, the collection of all *finitely representable sets of models in Λ* is given by: $\text{FR}(\Lambda) := \{\mathbb{M} \subseteq \mathfrak{M} \mid \exists \mathcal{B} \in \mathcal{P}_f(\mathcal{L}) : \text{Mod}(\mathcal{B}) = \mathbb{M}\}$. Using this notion we propose two constructions: $\text{MaxFRSubs}(\mathbb{M}, \Lambda)$, the \subseteq -maximal subsets of \mathbb{M} that are finitely representable in Λ ; and its analogous $\text{MaxFRSubs}(\mathbb{M}, \Lambda)$ which considers the \subseteq -minimal supersets instead. Next, we define two kinds of model change operations which take as input the current finite base \mathcal{B} and a set of input models \mathbb{M} : *reception* ($\text{rcp}(\mathcal{B}, \mathbb{M})$) when we want to accept the input models; and *eviction* ($\text{evc}(\mathcal{B}, \mathbb{M})$) when we want to reject them instead. Besides providing the constructions, we also identify the rationality postulates that characterise these functions.

In propositional logic, representing the body of knowledge as a finite set of formulae is straightforward. In fact, we can define eviction and reception quite easily if the signature is finite in this case. However, this may not be so in other logics, such as Description Logics (DLs) [Baader et al., 2007]. This problem emerges because there are satisfaction systems in which \mathbb{M} either $\text{MinFRSups}(\mathbb{M}, \Lambda) = \emptyset$ or $\text{MaxFRSubs}(\mathbb{M}, \Lambda) = \emptyset$, for some set of models \mathbb{M} .

We then consider the case of the DL $\mathcal{ALC}_{\text{bool}}$, which corresponds to the traditional DL \mathcal{ALC} enriched with boolean connectives between formulas. We prove that the satisfaction system of $\mathcal{ALC}_{\text{bool}}$ is not compatible with reception. As a workaround, we provide new constructions,

exploiting the available boolean connectives to define eviction and reception operations that are adequate to this particular satisfaction system. More specifically, we use the notion of quasimodels Agi et al. [2003] to normalise bases in $\mathcal{ALC}_{\text{bool}}$ into DNFs and perform eviction by removing disjuncts and reception by adding disjuncts to the result. Finally, we identify the properties that these new functions satisfy in terms of rationality postulates.

In conclusion, we introduce a new paradigm of Belief Change by considering sets of models as input and enforcing the finite representability of an agent’s epistemic state. We also define and characterise two operations, eviction and reception, which mimic their traditional counterparts in Belief Change, contraction and expansion. Furthermore, we identify some limitations of the constructions proposed and studied one concrete case in which some adaptations were needed. This work opens numerous research directions, including the design of an operation analogous to the traditional revision operations, which avoids inconsistent states and connections with recent approaches in Belief Change and Ontology Repair that also allow rewritings of the initial base to preserve more information.

Acknowledgements

Guimarães is supported by the ERC project LOPRE (819416), led by Prof Saket Saurabh. Ozaki is supported by the Research Council of Norway, project number 316022. S. Ribeiro is supported by the German Research Association (DFG), project number 424710479. Parts of this work have been done in the context of CEDAS (Center for Data Science, University of Bergen, Norway).

References

- Kurucz Agi, Frank Wolter, Michael Zakharyashev, and Dov Gabbay. *Many-dimensional modal logics : theory and applications*. Elsevier North Holland, Amsterdam Boston, 2003. ISBN 0444508260.
- Marc Aiguier, Jamal Atif, Isabelle Bloch, and Céline Hudelot. Belief revision, minimal change and relaxation: A general framework based on satisfaction systems, and applications to description logics. *Artificial Intelligence*, 256:160–180, mar 2018. doi: 10.1016/j.artint.2017.12.002.
- Carlos E. Alchourrón, Peter Gärdenfors, and David Makinson. On the Logic of Theory Change: Partial Meet Contraction and Revision Functions. *Journal of Symbolic Logic*, 50(2):510–530, 1985.
- Franz Baader, Diego Calvanese, Deborah McGuinness, Daniele Nardi, and Peter Patel-Schneider, editors. *The Description Logic Handbook: Theory, Implementation, and Applications*. Cambridge University Press, second edition, 2007.
- James P. Delgrande, Pavlos Peppas, and Stefan Woltran. General belief revision. *J. ACM*, 65(5):29:1–29:34, 2018.
- Sven Ove Hansson. *A Textbook of Belief Dynamics: Theory Change and Database Updating*. Applied Logic Series. Kluwer Academic Publishers, 1999.

Introducing Proof Tree Automata and Proof Tree Graphs

Valentin D. Richard
LORIA, UMR 7503
Université de Lorraine, CNRS, Inria
54000 Nancy, France
valentin.richard@loria.fr

Introduction Research in structural proof theory [7] may lead to considering large calculi, containing several dozens of rules (e.g. 68 rules found in [5]). Keeping track of all possible combinations of these rules is an issue. This problem is particularly critical at the design phase, when trying to come up with a calculus which meets some desiderata.

When trying to design a calculus, researchers often do not just want to test *whether* it has the expected specification, but to know *why* and *how* it does or does not. Their requirements often revolve around intuitions about connectives and rules, e.g. “What happens if we add or remove this rule?”.

The combinatorics of rules also brings a challenge at proof phase, when trying to demonstrate properties about a calculus. Many theorems on calculi still make use of case disjunction. Such a strategy becomes difficult and fastidious as the size of the system increases. There is a desire to get a larger picture of calculi, to get new insights about them.

Approaches based on graphical languages, like proof nets or string diagrams, turned out to be of great use to give visual intuitions. Nevertheless, they often focus on a single derivation and divert from the very structure of derivation trees.

Proposal The contribution of this article is twofold:

1. Introducing a novel graphical representation of a calculus aiming at bringing better intuition about the interconnection of rules and sequents
2. Providing a new perspective of proof search through tree automata theory

Proof tree graphs The graphical representation we introduce is called Proof Tree Graph (PTG). A PTG can represent a calculus, or more generally any term deduction system.¹

- Vertices are sets of terms (e.g. sequents, if we work with a sequent calculus)
- Edges are rules

In Fig. 1, we display a PTG for implicative sequent calculus ImpL ((1) in appendix), and in Fig. 2 a PTG for the sequent calculus of the multiplicative fragment of linear logic (MLL, see [3]).² Edge $\Delta, \varphi \vdash \psi \xrightarrow{\rightarrow I} \Delta \vdash \varphi \rightarrow \psi$ represents rule $(\rightarrow I)$, from the hypothesis to the conclusion. The axiom is

¹Technically, a PTG is a directed hypergraph [1, chap. 6] with additional dashed edges.

²In both calculi considered, we take commas to be multiset separators.

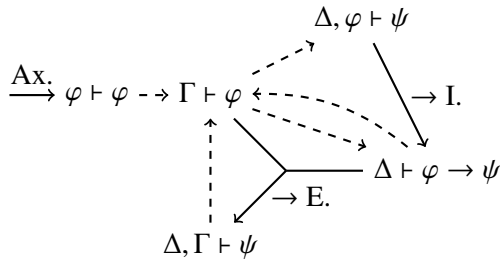


Figure 1: Proof Tree Graph for implicational sequent calculus.

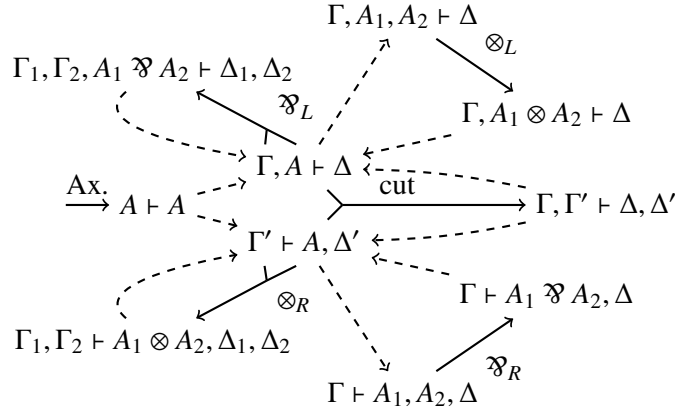


Figure 2: Proof Tree Graph for MLL.

represented by an edge with no source as it has no hypothesis. By the same idea, rule $\rightarrow E.$ has two source vertices as it has two hypotheses.

A dashed edge $u \dashrightarrow v$ means that we can pass from vertex u to v without applying a rule. For example, $\varphi \vdash \varphi \dashrightarrow \Gamma \vdash \varphi$ means that both vertices share a common instance sequent, e.g. $p \vdash p$ if p is an atomic formula. Note here that a sequent actually stands for a set of instances. They are thus taken up to meta-variable renaming.

The goal of a PTG is to give visual intuitions about the relationships between rules by linking the hypotheses and the conclusions of these rules. This way, it appears clearly how certain rules can follow other rules. Thus, a PTG illustrates the whole system, and not a particular derivation. For example, the antecedent – succedent symmetry of linear logic is visible through the horizontal axis symmetry in Fig. 2.

One recipe to create a PTG out of any calculus \mathcal{K} is the following:

1. As vertex, take any hypothesis or conclusion of a rule of \mathcal{K}
2. Create an n -ary edge for every n -ary rule on the corresponding vertices
3. Add a dashed edge $u \dashrightarrow v$ for every vertices u and v which share an instance (i.e. $u \cap v \neq \emptyset$)

Proof nets [4] and string diagrams [8], widespread graphical languages, differ from PTGs on the kind of object represented. They can only represent (sets of equivalent) derivations, whereas PTGs allow us to represent the whole calculus. Therefore, PTGs give an overarching image of the rules. Some first idea of such a diagram of rules can be found in [5, Fig. 2].

Proof Tree Automata If all rules of a calculus are unary, a PTG on that calculus looks like the graphical representation of a non-deterministic finite automaton: vertices are states and edges are transitions. In this setting, axiom targets make initial states, dashed edges are ε -transitions and all states are accepting.

We build on that analogy to retro-engineer a new kind of tree automata called Proof Tree Automata (PTA), which graphical representations are PTGs. A PTA \mathcal{A} on a calculus \mathcal{K} is a tree automaton ([2]) with additional material. Its language is the derivation language of \mathcal{K} . A forward proof-search in \mathcal{K} corresponds to a bottom-up run in \mathcal{A} .

The additional material is a pair of relations called control relations. Their goal is to ensure that, while parsing a proof tree, hypothesis terms and conclusion terms are correctly related.

Using automata and graphs is an open door to topological methods for term deduction. One goal of PTA and PTGs is to provide a tool with which we can translate properties expressed on sets of derivation trees into properties expressed on automaton runs or graph walks.

Additional results and open questions An interesting point is the comparison of a PTA \mathcal{A} on a calculus \mathcal{K} and its tree automaton counterpart $F(\mathcal{A})$, i.e. with control relations removed. The language of $F(\mathcal{A})$ is wider than the language of \mathcal{A} because it contains derivations which are not correct wrt. \mathcal{K} .

One can build a function³ U from \mathcal{K} to $F(\mathcal{A})$, mapping sets of terms to states and derivations to runs. Function U has the following property: a derivations of $F(\mathcal{A})$ is correct iff it belong to the image of U . Thus, a PTA appears as tree automaton parameterized by a calculus.

As a novel tool, many questions arise about PTA and PTGs. Particularly, we deem investigations about relations on PTA to be relevant. When can we say that a PTA is finer than another one? Could we design a criterion for PTA equivalence (i.e. having the same language). It would also be useful to find graph rewriting techniques to compute these problems on PTGs.

References

- [1] Alain Bretto. *Hypergraph Theory: An Introduction*. Springer Science & Business Media, April 2013.
- [2] Hubert Comon, Max Dauchet, Rémi Gilleron, Florent Jacquemard, Denis Lugiez, Christof Löding, Sophie Tison, and Marc Tommasi. *Tree Automata Techniques and Applications*. 2008.
- [3] Roberto Di Cosmo and Dale Miller. Linear logic. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, summer 2019 edition, 2019.
- [4] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.
- [5] Giuseppe Greco, Valentin D. Richard, Michael Moortgat, and Apostolos Tzimoulis. Lambek-Grishin calculus: Focusing, display and full polarization. *Logic and Structure in Computer Science and Beyond*, 2021.
- [6] Paul-André Melliès and Noam Zeilberger. Functors are Type Refinement Systems. In *42nd ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL 2015)*, January 2015.
- [7] Sara Negri and Jan von Plato. *Structural Proof Theory*. Cambridge University Press, Cambridge, 2001.
- [8] Peter Selinger. A Survey of Graphical Languages for Monoidal Categories. In Bob Coecke, editor, *New Structures for Physics*, Lecture Notes in Physics, pages 289–355. Springer, Berlin, Heidelberg, 2011.

Sequent calculus for implicational logic

$$\frac{}{\varphi \vdash \varphi} \text{Ax.} \quad \frac{\Delta, \varphi \vdash \psi}{\Delta \vdash \varphi \rightarrow \psi} \rightarrow \text{I.} \quad \frac{\Delta \vdash \varphi \rightarrow \psi \quad \Gamma \vdash \varphi}{\Delta, \Gamma \vdash \psi} \rightarrow \text{E.} \quad (1)$$

³Actually, U is a monoidal functor between monoidal categories. This way, U is a refinement system [6].

On extracting variable Herbrand disjunctions

Andrei Sipos^{a,b}

^aResearch Center for Logic, Optimization and Security (LOS), Department of Computer Science,
Faculty of Mathematics and Computer Science, University of Bucharest,
Academiei 14, 010014 Bucharest, Romania

^bSimion Stoilow Institute of Mathematics of the Romanian Academy,
Calea Griviței 21, 010702 Bucharest, Romania

Web address: <https://cs.unibuc.ro/~asipos/>

Email address: andrei.sipos@fmi.unibuc.ro

In 2005, Gerhardy and Kohlenbach [1] presented a novel method to prove classical Herbrand theorem for first-order logic, namely by using Gödel's *Dialectica* interpretation, in particular a variant inspired by that of Shoenfield [3], to construct witnesses that realize the interpreted formulas in a system similar to Gödel's T, but lacking recursors, and having case distinction functionals added in order to realize contraction. For a formula of the form $\exists x\varphi$ as in the usual statement of Herbrand's theorem, the extracted term is then β -reduced and it is shown that the resulting term has a sufficiently well-behaved form that one can read off it the classical Herbrand terms.

The *Dialectica* interpretation usually serves as a loose analogue to Herbrand's theorem to systems which include arithmetical axioms; and in a highly sophisticated form, it plays nowadays a central role in the research program of *proof mining*, given maturity by the school of Kohlenbach (see e.g. his book [2]), a program which aims to apply term extraction theorems to ordinary mathematical proofs in order to uncover information that may be not immediately apparent. One striking feature of this sort of proof interpretations is that they generally do not extract terms expressible in the original system under discussion, but usually go beyond it in that they make use of concepts like higher-type functionals or recursion along large countable ordinals.

Despite this fact, it has been observed that in certain basic endeavours of proof mining, the extracted terms may take the form of a classical Herbrand disjunction but of variable length, and what we do here is to attempt a logical explanation of this empirical fact, and towards that end, we extend the proof of Gerhardy and Kohlenbach to theories which are on the level of first-order arithmetic, dealing with the corresponding recursors (which one generally uses to interpret induction) by using Tait's infinite terms [5]. This passage to the infinite allows us to prove in this extended context the corresponding version of the well-behavedness property mentioned before. We also illustrate our result with a proof mining case study.

The results presented in this talk may be found in [4].

References

- [1] P. GERHARDY, U. KOHLENBACH, *Extracting Herbrand disjunctions by functional interpretation*, *Archive for Mathematical Logic*, vol. 44 (2005), no. 5, pp. 633–644.
- [2] U. KOHLENBACH, *Applied proof theory: Proof interpretations and their use in mathematics*, Springer Monographs in Mathematics, Springer-Verlag, 2008.
- [3] J. SHOENFIELD, *Mathematical Logic*, Addison-Wesley Series in Logic, Addison-Wesley Publishing Co., 1967.
- [4] A. SIPOŞ, *On extracting variable Herbrand disjunctions*, arXiv:2111.12133 [math.LO], 2021. To appear in: *Studia Logica*.
- [5] W. W. TAIT, *Infinitely long terms of transfinite type*, *Formal Systems and Recursive Functions* (J. N. Crossley and M. A. E. Dummett, editors), Elsevier, Amsterdam, 1965, pp. 176–185.

On Various Weak First-Order Theories

Lars Kristiansen^{1,2}

¹ Department of Mathematics, University of Oslo, Norway (larsk@math.uio.no)

² Department of Informatics, University of Oslo, Norway

I will survey some research we (Juvenal Murwanashyaka and myself) have done in Oslo during the last few years. My talk will not be very technical, and it should be accessible to a broad audience.

Concatenation Theory. First-order concatenation theory can be compared to first-order number theory, e.g., Peano Arithmetic or Robinson Arithmetic. The universe of a standard structure for first-order number theory is the set of natural numbers. The universe of a standard structure for first-order concatenation theory is the set of finite strings over some alphabet. A first-order language for number theory normally contains two binary functions symbols. In a standard structure these symbols will be interpreted as addition and multiplication. A first-order language for concatenation theory normally contains just one binary function symbol. In a standard structure this symbol will be interpreted as the operator that concatenates two strings. A classical first-order language for concatenation theory—like e.g. the ones studied in Quine [7] and Grzegorzczuk [8]—contains no other non-logical symbols apart from constant symbols.

We extend the language of classical concatenation theory with a binary relation symbol \sqsubseteq . The relation \sqsubseteq might be interpreted as, e.g., (i) x is a substring of y , (ii) x is a prefix of y , (iii) x is shorter than y . This makes it possible to introduce bounded quantifiers (similar to those we know from first-order number theory). In [1] and [2], we axiomatize first-order concatenation theory with \sqsubseteq . We give a number of axiomatizations with various desirable properties, we prove normal-form results, and moreover, we prove a number of decidability and undecidability results.

Our axiomatization of concatenation theory induces a number of weak first-order theories. In [3] and [4], we compare the strength of various first-order concatenation theories to the strength of other weak theories known from the literature, like e.g., Robinson's Q, Robinson's R and Grzegorzczuk's TC, by proving interpretability results (I will explain what it means that a theory T is interpretable in a theory S).

Term Theory. In [5], we introduce two first-order theories, WT and T, over the language $\mathcal{L}_T = \{\perp, \langle \cdot, \cdot \rangle, \sqsubseteq\}$ where \perp is a constant symbol, $\langle \cdot, \cdot \rangle$ is a binary function symbol and \sqsubseteq is a binary relation symbol. The intended model for these theories is a term model: The universe is the set of all variable-free \mathcal{L}_T -terms. Each term is interpreted as itself, and \sqsubseteq is interpreted as the subterm relation (s is a subterm of t iff $s = t$ or $t = \langle t_1, t_2 \rangle$ and s is a subterm of t_1 or t_2). This

$$\begin{array}{l}
T_1 \quad \forall xy[\langle x, y \rangle \neq \perp] \\
T_2 \quad \forall x_1 x_2 y_1 y_2[\langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle \rightarrow (x_1 = y_1 \wedge x_2 = y_2)] \\
T_3 \quad \forall x[x \sqsubseteq \perp \leftrightarrow x = \perp] \\
T_4 \quad \forall xyz[x \sqsubseteq \langle y, z \rangle \leftrightarrow (x = \langle y, z \rangle \vee x \sqsubseteq y \vee x \sqsubseteq z)].
\end{array}$$

Fig. 1. The axioms of T .

model might also be seen as a model where the universe is the set of all finite full binary trees and \sqsubseteq is the subtree relation.

In [5], we prove that WT is mutually interpretable with Robinson's R , and moreover, we prove that T is interpretable in Robinson's Q and conjecture that Robinson's Q is interpretable in T . This conjecture has later been proved by Damnjanovic [6].

Figure 1 shows the axioms of T . For more on Q , R and interpretability in general, see Tarski [9].

References

1. Kristiansen L., Murwanashyaka J. (2018) *Decidable and Undecidable Fragments of First-Order Concatenation Theory*. In: Manea F., Miller R., Nowotka D. (eds) *Sailing Routes in the World of Computation*. CiE 2018. Lecture Notes in Computer Science, vol 10936. Springer, Cham. https://doi.org/10.1007/978-3-319-94418-0_25
2. Kristiansen, L., Murwanashyaka, J. *First-Order Concatenation Theory with Bounded Quantifiers*. Arch. Math. Logic **60** (2021), 77–104. <https://doi.org/10.1007/s00153-020-00735-6>
3. Murwanashyaka, J. *Weak essentially undecidable theories of concatenation*. Arch. Math. Logic (2022). <https://doi.org/10.1007/s00153-022-00820-y>
4. Murwanashyaka, J. *Weak Essentially Undecidable Theories of Concatenation II*. arXiv:2112.13907 [math.LO] <https://doi.org/10.48550/arXiv.2112.13907>
5. Kristiansen L., Murwanashyaka J. (2020). *On Interpretability Between Some Weak Essentially Undecidable Theories*. In: Anselmo M., Della Vedova G., Manea F., Pauly A. (eds). *Beyond the Horizon of Computability*. CiE 2020. Lecture Notes in Computer Science, vol 12098. Springer, Cham. https://doi.org/10.1007/978-3-030-51466-2_6
6. Damnjanovic, Z. *Mutual Interpretability of Weak Essentially Undecidable Theories*. J.Symb.Log. (2022), 1-50. doi:10.1017/jsl.2022.15
7. Quine, W. V. *Concatenation as a Basis for Arithmetic*. J.Symb.Log. **11** (1946),105-114.
8. Grzegorzcyk, A. *Undecidability without Arithmetization*. Stud. Log. **79** (2005),163-230.
9. Tarski, A., Mostowski, A., Robinson, R. M. *Undecidable Theories*. North-Holland, Amsterdam (1953).

Logic of Sentential Operators

Michał Walicki

Department of Informatics,
University of Bergen, Norway

Michal.Walicki@uib.no

Extending FOL with quantification over all sentences (of the extended language) does not increase expressive power, hence, does not yield any paradoxes. Further extension with sentential predicates, by a form of definitional extension, does not lead to any paradoxes, either. Note that we consider predicates on sentences, not their names or codes; such predicates are often referred to as operators. For instance, every FOL theory has a conservative extension with the operator axiomatized by the single sentence (T) $\forall\phi(\mathcal{T}\phi \leftrightarrow \phi)$, [8]. This exemplifies the arising circularity, since sentence (T) provides a possible instantiation of ϕ . Semantics, handling adequately such circularity, is obtained by extending a presentation of the standard semantics of FOL using kernels of digraphs [7]. Analogous extensions can be obtained for higher-order classical logic.

Axiom (T) gives a trivial example of a sentential operator. Introducing such by definitional extensions, like (T), retains consistency, but once we allow arbitrary definitions, paradoxes can appear. For instance, John can say to be sometimes lying, $J(\neg\forall\phi(J\phi \rightarrow \phi))$. Reasoning system LSO, extending LK with two rules, allows then to deduce that he indeed does. If he does not say anything else, LSO derives that his claim is paradoxical, implying a contradiction.

Following [6], kernel semantics is refined to semikernels, giving a paraconsistent interpretation of such theories which, being locally coherent, lead however to a contradiction and have no classical models. Paradoxes arise only in the metalanguage but, typically, do not affect truth values of most other statements. In particular, object language can always be interpreted consistently, independently from the possible confusion caused at the metalevel by the unfortunate statements – generally, by unfortunate valuations of sentential predicates, like J . Semikernel semantics, with the complete reasoning in LSO, determine then part of the language involved in the paradox. John’s paradox, as the informal liar, entails in LSO that he tells the truth iff he is lying, but nothing about other persons’ or object level statements.

Extension of LSO with (cut) becomes sound and complete for the explosive kernel semantics, where paradoxes entail not only specific contradictions but everything. LSO is thus an example of a non-transitive logic, where (cut) is not admissible. However, it differs significantly from other examples of such logics, e.g., [4, 5], that are trivialized by (cut). In our case, (cut) turns LSO into an essentially classical logic, exploding only on inconsistent theories. For the consistent ones, it remains sound and its restriction to the FOL language still coincides with classical logic.

Sentential operators allow axiomatic specializations, for instance, to various modal logics, and analysis of paradoxes of intensional type. They are, however, represented and analysed in the same way as semantic paradoxes. The liar, saying only “Every sentence I am saying now is false”, is not significantly different from John not believing any of his beliefs nor from a club whose members are all people not belonging to any club. Problems are caused by the same patterns, typically, of vicious circularity, captured in our graph-based semantics by odd cycles. Other patterns are possible and also Yablo paradox instantiates our general notion.

LSO allows thus for self-reference and paradoxes without arithmetizing metalanguage or internalizing it by other means in object language. When metalanguage is so arithmetized, the source of paradoxes becomes convention (T), which therefore has to be restricted. Representing sentential operators by predicates over arithmetized syntax is neither obvious nor innocent, [3].

Montague, [2], showed that arithmetized syntax with a predicate capturing a few modal properties yields paradox, unlike the corresponding modal logic utilizing sentential operators. This problem, caused by diagonalization lemma, appears already when operators like negation or implication are attempted modelled by predicates on sentence codes, [1]. LSO replaces therefore arithmetized syntax by sentential operators, and identifies their definitions as the only source of paradoxes. It marks the first step on the planned way to avoid paradoxes by maintaining syntactic distinction between metalanguage and object-language and observing the form of definitions of sentential operators.

References

- [1] H. Deutsch. Diagonalization and truth functional operators. *Analysis*, 70(2):215–217, 2010.
- [2] R. Montague. Syntactical treatment of modality, with corollaries on reflection principles and finite axiomatizability. *Acta philosophica Fennica*, 16:153–167, 1963.
- [3] W. N. Reinhardt. Necessity predicates and operators. *Journal of Philosophical Logic*, 9(4):437–450, 1980.
- [4] D. Ripley. Conservatively extending classical logic with transparent truth. *The Review of Symbolic Logic*, 5(2):354–378, 2012.
- [5] D. Ripley. Paradoxes and failures of cut. *Australasian Journal of Philosophy*, 91(1):139–164, 2013.
- [6] M. Walicki. Resolving infinitary paradoxes. *Journal of Symbolic Logic*, 82(2):709–723, 2017.
- [7] M. Walicki. Extensions in graph normal form. *Logic Journal of the IGPL*, 30(1):101–123, February 2022.
- [8] M. Walicki. Logic of sentential predicates. 2022. [in preparation, www.ii.uib.no/~michal/LSP.pdf].

Weak Essentially Incomplete Theories of Concatenation

Juvenal Murwanashyaka

Department of Mathematics, University of Oslo, Norway

A countable first-order theory is called *essentially undecidable* if any consistent extension, in the same language, is undecidable (there is no algorithm for deciding whether an arbitrary sentence is a theorem). A countable first-order theory is called *essentially incomplete* if any recursively axiomatizable consistent extension is incomplete. It is known that a theory is essentially undecidable if and only if it is essentially incomplete (see for example Chapter 1 of Tarski et al. [6]). Two theories that are known to be essentially undecidable are Robinson arithmetic Q and the related theory R (see Figure 1 for the axioms of R and Q).

The theory of concatenation TC was introduced by Grzegorzczuk in [2] where he also showed that it is undecidable (see Figure 2 for the axioms of TC). The language of TC consists only of two constant symbols 0, 1 and a binary function symbol \circ . The intended model of TC is a free semigroup with at least two generators. Grzegorzczuk's motivation for introducing TC was that, as computation involves manipulation of text, the notion of computation can be formulated on the basis of discernibility of text without reference to natural numbers. Then, undecidability of first-order logic and essential undecidability can be explained using a theory of strings thereby avoiding complicated coding of syntax based on natural numbers. In [3], Grzegorzczuk and Zdanowski showed that TC is essentially undecidable. This was further improved in Ganea [1], Visser [7] and Švejdar [5] where it was shown that TC is mutually interpretable with Robinson arithmetic.

In this talk, which is based on Murwanashyaka [4], we show that two theories, WD and D, are mutually interpretable with R and Q, respectively (see Figure 3 for the axioms of WD and D). The theories WD and D are theories in the language of TC extended with a binary relation symbol \preceq . The intended structure \mathfrak{D} is the free semigroup with two generators extended with the prefix relation which we denote $\preceq^{\mathfrak{D}}$. The theories WD and D are purely universally axiomatised,

The Axioms of R	The Axioms of Q
$R_1 \quad \bar{n} + \bar{m} = \overline{n + m}$	$Q_1 \quad \forall xy [x \neq y \rightarrow Sx \neq Sy]$
$R_2 \quad \bar{n} \times \bar{m} = \overline{n \times m}$	$Q_2 \quad \forall x [Sx \neq 0]$
$R_3 \quad \bar{n} \neq \bar{m} \quad \text{if } n \neq m$	$Q_3 \quad \forall x [x = 0 \vee \exists y [x = Sy]]$
$R_4 \quad \forall x [x \leq \bar{n} \rightarrow \bigvee_{k \leq n} x = \bar{k}]$	$Q_4 \quad \forall x [x + 0 = x]$
$R_5 \quad \forall x [x \leq \bar{n} \vee \bar{n} \leq x]$	$Q_5 \quad \forall xy [x + Sy = S(x + y)]$
	$Q_6 \quad \forall x [x \times 0 = 0]$
	$Q_7 \quad \forall xy [x \times Sy = (x \times y) + x]$

Figure 1: Non-logical axioms of R and Q.

The Axioms of TC

$$\begin{aligned}
\text{TC}_1 & \forall xyz [x(yz) = (xy)z] \\
\text{TC}_2 & \forall xyzw [(xy = zw \rightarrow ((x = z \wedge y = w) \vee \\
& \qquad \exists u [(z = xu \wedge uw = y) \vee (x = zu \wedge uy = w)]))] \\
\text{TC}_3 & \forall xy [xy \neq 0] \\
\text{TC}_4 & \forall xy [xy \neq 1] \\
\text{TC}_5 & 0 \neq 1
\end{aligned}$$

Figure 2: Non-logical axioms of TC.

The Axioms of WD

$$\begin{aligned}
\text{WD}_1 & \bar{\alpha} \bar{\beta} = \overline{\alpha\beta} \\
\text{WD}_2 & \bar{\alpha} \neq \bar{\beta} \quad \text{if } \alpha \neq \beta \\
\text{WD}_3 & \forall x [x \preceq \bar{\alpha} \leftrightarrow \bigvee_{\gamma \preceq \bar{\alpha}} x = \bar{\gamma}]
\end{aligned}$$

The Axioms of D

$$\begin{aligned}
\text{D}_1 & \forall xyz [(xy)z = x(yz)] \\
\text{D}_2 & \forall xy [x \neq y \rightarrow (x0 \neq y0 \wedge x1 \neq y1)] \\
\text{D}_3 & \forall xy [x0 \neq y1] \\
\text{D}_4 & \forall x [x \preceq 0 \leftrightarrow x = 0] \\
\text{D}_5 & \forall x [x \preceq 1 \leftrightarrow x = 1] \\
\text{D}_6 & \forall xy [x \preceq y0 \leftrightarrow (x = y0 \vee x \preceq y)] \\
\text{D}_7 & \forall xy [x \preceq y1 \leftrightarrow (x = y1 \vee x \preceq y)]
\end{aligned}$$

Figure 3: Non-logical axioms of WD and D.

in contrast to Q and TC which have the Π_2 -axioms Q_3 , TC_2 .

References

- [1] Ganea, M.: Arithmetic on semigroups. *Journal of Symbolic Logic* 74, 265-278 (2009)
- [2] Grzegorzcyk, A.: Undecidability without Arithmetization. *Studia Logica* 79, 163–230 (2005)
- [3] Grzegorzcyk, A., Zdanowski, K.: Undecidability and concatenation. In: Ehrenfeucht et al., “Andrzej Mostowski and Foundational Studies”, pp. 72-91, IOS, Amsterdam, 2008.
- [4] Murwanashyaka, J.: Weak Essentially Undecidable Theories of Concatenation. *Archive for Mathematical Logic*, 2022, <https://doi.org/10.1007/s00153-022-00820-y>
- [5] Švejdar, V.: On Interpretability in the Theory of Concatenation. *Notre Dame Journal of Formal Logic* 50, 87-95 (2009)
- [6] Tarski, A., Mostowski, A., Robinson, R. M.: *Undecidable theories*. North-Holland, Amsterdam (1953).
- [7] Visser, A.: Growing commas. A study of sequentiality and concatenation. *Notre Dame Journal of Formal Logic* 50, 61-85 (2009)

Generalizing Epistemic Updates

Igor Sedlár

Czech Academy of Sciences, Institute of Computer Science

A generalization of Public Announcement Logic where epistemic modalities may be weaker than S5-style knowledge operators and where announcements may not be truthful is provided by the framework of *epistemic updates*. In this framework, an update of the epistemic state of an agent a with a proposition $X \subseteq S$ transforms the model so that the set $E_a(s)$ of epistemically accessible states from any state $s \in S$ is replaced by

$$E_a^X(s) = E_a(s) \cap X. \quad (1)$$

That is, the model transformation representing the update does not erase non- X states, it only erases a 's epistemic accessibility arrows leading to non- X states. The effects of epistemic update are expressed by means of a binary update operator $[\]$ with the usual semantic clause

$$(M, s) \models [\varphi]\psi \iff (M^{\llbracket\varphi\rrbracket_M}, s) \models \psi, \quad (2)$$

where $\llbracket\varphi\rrbracket_M = \{s \mid (M, s) \models \varphi\}$ and, for all epistemic models $M = (S, \{E_a \mid a \in Ag\}, V)$ and $X \subseteq S$, $M^X = (S, \{E_a^X \mid a \in Agt\}, V)$. A prominent example of this approach is Gerbrandy and Groeneveld's [2]; see also [5, 6, 11].

In this paper we study further generalizations of this framework, based on replacing the intersection operation in (1) with an *arbitrary binary update operation* $\otimes : 2^S \times 2^S \rightarrow 2^S$. The motivation of such a generalization is to model epistemic updates that may not be truthful but, in addition, may not be accepted by all agents equally or may cause agents to renounce some of their previously held beliefs. For instance, the update operation \otimes may be set up in such a way that $E_a(s) \otimes X$ is not a subset of $E_a(s)$ (i.e. the agent a “looses information”) nor a subset of X (i.e. the announcement is not accepted). A natural setting where such generalized epistemic updates occur is *public argumentation*, that is, putting forward arguments (represented by $X \subseteq S$) that may not be truthful, may not persuade all agents equally, but may cause some agents to change their mind. For example, a good argument in favour of a piece of legislation may persuade a member of the parliament to change their negative attitude towards the piece of legislation. On the other hand, an argument based on conspiracy theories may be ignored by some MPs and, unfortunately, accepted by others.

Our basic semantic framework extends epistemic models with a ternary accessibility relation R , familiar from semantics of relevant and other substructural logics [8, 9], representing \otimes via

$$X \otimes Y = \{s \mid \exists t, u (Rtus \ \& \ t \in X \ \& \ u \in Y)\}. \quad (3)$$

On the syntactic side, we extend the standard language of PAL with a relevant implication connective \rightarrow corresponding to R : $\varphi \rightarrow \psi$ is satisfied in s iff $Rstu$ implies that φ is satisfied by t only if ψ is satisfied by u . Hence, $\llbracket\varphi \rightarrow \psi\rrbracket_M = \{s \mid \{s\} \otimes \llbracket\varphi\rrbracket_M \subseteq \llbracket\psi\rrbracket_M\}$.

Building on our earlier work [7, 10] we prove a soundness and completeness result for the basic logic of this framework using reduction axioms. The reduction axiom for the epistemic modality \Box_a , namely

$$[\varphi]\Box_a\psi \equiv \Box_a(\varphi \rightarrow [\varphi]\psi) \quad (4)$$

uses the relevant implication connective \rightarrow . We stress that our logic GPAL is non-classical only to a limited degree: it extends normal modal logic based on classical propositional logic with a relevant implication connective \rightarrow and the generalized epistemic update operator $[\]$. The relevant implication connective is associated with the generalized update operation \otimes and allows to formulate a reduction axiom for $[\]$; the underlying propositional logic is classical.

A further generalization of the framework takes R to be a function from the set of agents to ternary relations on S , along with introducing agent-indexed implication operators \rightarrow_a . This generalization is related to Group Announcement Logic [12], where formulas of the form $\langle G \rangle \varphi$ express that there is an announcement of formulas describing the epistemic states of agents in G , say $\bigwedge_{i \in G} K_i \psi_i$, such that the announcement makes φ true. In our setting, this idea is generalized in two ways. First, we allow the update with *the same* formula announced by two different agents to have different effects. Second, the epistemic action at hand is not necessarily a S5-style public announcement. Also, our update operator $[\varphi]_a$ does not quantify over announcements. Parametrizing the update operation \otimes with agents allows to fine-tune the effects of the update depending on who the announcing agent is. In realistic settings, the effects of updates (e.g. arguments) usually depend on who the source of the update is.

As expected given our informal interpretation of generalized updates in terms of argumentation, we will also show that our framework has interesting links to abstract argumentation theory [1]. Grossi has shown that modal logic can formalize notions of abstract argumentation theory [3, 4]; this follows from the fact that abstract argumentation frameworks are Kripke frames. In contrast to this approach, we will represent arguments not as nodes in a Kripke frame, but as formulas. Most importantly, we will represent the claim that φ attacks ψ as

$$\varphi \hookrightarrow \psi := \varphi \rightarrow [\varphi]\neg\psi. \quad (5)$$

Using (5) we may express that a finite set of formulas (arguments) is *conflict-free* and, consequently, we may express argumentation-theoretic notions such as acceptability and admissibility of (finite) sets of arguments and various notions of extension central to abstract argumentation theory. This approach to modelling abstract argumentation has two advantages. First, representing arguments by formulas allows to articulate the internal structure of arguments (e.g. if φ attacks ψ , then $\chi \wedge \varphi$ attacks ψ). Second, our framework allows to express epistemic attitudes of agents towards attack relations between arguments ($\Box_a(\varphi \hookrightarrow \psi)$, “ a believes that φ attacks ψ ”, equivalent to $[\varphi]\Box_a\neg\varphi$ via (4)) and towards argumentation-theoretic properties of sets of arguments, and also effects of updates/announcements regarding these relations and properties. These attitudes are essential in *strategic argumentation*: information about the attack relations recognized by agent a play a role if agent b aims at persuading a to accept a specific proposition.

Acknowledgement. The author is grateful to SLSS 2022 reviewers for valuable suggestions, and to Vít Punčochář and Andrew Tedder for discussions on this work in progress.

References

- [1] P. M. Dung. On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games. *Artificial Intelligence*, 77(2):321–357, 1995.
- [2] J. Gerbrandy and W. Groeneveld. Reasoning about information change. *Journal of Logic, Language and Information*, 6(2):147–169, Apr 1997.
- [3] D. Grossi. On the logic of argumentation theory. In W. van der Hoek, G. Kaminka, Y. Lespérance, and S. Sen, editors, *Proc. 9th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2010)*, pages 409–416, 2010.
- [4] D. Grossi. Argumentation in the view of modal logic. In P. McBurney, I. Rahwan, and S. Parsons, editors, *Proc. Int. Workshop on Argumentation in Multi-Agent Systems (ArgMAS 2010)*, pages 190–208, Berlin, Heidelberg, 2011. Springer Berlin Heidelberg.
- [5] B. Kooi and B. Renne. Arrow update logic. *Review of Symbolic Logic*, 4(4):536–559, 2011.
- [6] F. Landman. *Towards a Theory of Information. The Status of Partial Objects in Semantics*. Groningen-Amsterdam Studies in Semantics, volume 6 (GRASS 6). Foris Publications, Dordrech and Riverton, New Jersey, 1986.
- [7] V. Punčochář and I. Sedlár. Relevant epistemic logic with public announcements and common knowledge. In P. Baroni, C. Benz Müller, and Y. N. Wáng, editors, *Logic and Argumentation. Proc. of the 4th International Conference on Logic and Argumentation (CLAR 2021)*, volume 13040 of *Lecture Notes in Computer Science*, pages 342–361, Cham, 2021. Springer International Publishing.
- [8] G. Restall. *An Introduction to Substructural Logics*. Routledge, London, 2000.
- [9] R. Routley and R. K. Meyer. Semantics of entailment. In H. Leblanc, editor, *Truth, Syntax and Modality*, pages 194–243. North Holland, Amsterdam, 1973.
- [10] I. Sedlár and A. Tedder. Situated epistemic updates. In S. Ghosh and T. Icard, editors, *Proc. 8th International Conference on Logic, Rationality, and Interaction (LORI 2021)*, pages 192–200, Cham, 2021. Springer International Publishing.
- [11] B. van Linder, W. van der Hoek, and J. Meyer. Tests as epistemic updates. In *Proc. 11th European Conference on Artificial Intelligence, ECAI’94*, pages 331–335. John Wiley and Sons, Inc., 1994.
- [12] T. Ågotnes, P. Balbiani, H. van Ditmarsch, and P. Seban. Group announcement logic. *Journal of Applied Logic*, 8(1):62–81, 2010.

Purity of Formal Proofs

Robin Martinot (Utrecht University)

Purity has a long history as an ideal of proof for mathematicians, tracing back to early writings of Aristotle (Detlefsen, 2008). It concerns the idea that a pure proof should only draw upon notions that belong to the content of the theorem. Impure proofs distinguish themselves from pure proofs by making use of concepts that are ‘extraneous’ to what a theorem is about. A traditional value of purity is that it allows us to “become familiar with the specific details of the subject of the theorem” (Lehet, 2021), while impurity is commonly valued for its ability to unify and generalize different disciplines of mathematics. Purity and impurity have also been related to other conceptual values of proof such as simplicity and explanatoriness (Arana, 2017; Iemhoff, 2017; Lange, 2019).

In this talk, we will focus on characterizing purity for formal proofs. This will contribute to a better formal understanding of informal notions, and can be seen as a case study for how proof-theoretic criteria can identify philosophically meaningful derivations. Previous accounts of purity generally aim to explain the practical manifestation of purity for informal proofs (Arana and Detlefsen, 2011; Baldwin, 2013; Kahle and Pulcini, 2017). Instead, Arana (2009) investigates the use of cut elimination as a property mechanically guaranteeing purity for formal proofs, but concludes that this does not accurately represent practical purity. Hence, our approach for characterizing purity for formal proofs actively incorporates the intuitions of mathematicians. In the process, this leads to a new understanding of purity of proof, that better accommodates the attitude of contemporary mathematics by preserving values of both traditional purity and impurity.

First, we interpret the content of a theorem as the range of mathematical material that a theorem concerns, as captured by a particular formal theory. The intuitive nature of purity is preserved by letting the selection of this theory be heavily inspired by mathematical intuitions. Syntactic derivations that start from the axioms of the relevant formal theory may then be considered pure. In order to make this purity guarantee as inclusive as possible with respect to the intuitions of mathematicians, a notion of equality for theories is desirable. This notion should incorporate formal theories that only differ from the pure theory in superficial ways, and that still represent the same mathematical content. Motivated by the latter requirement, we will consider all definitional extensions of this theory as equal choices for purity.

Second, we argue that formal theories that do not directly capture intuitively pure content can be restricted to surrogates of pure content. We will attribute a secondary sense of purity to proofs in these restricted parts, since an informal theorem can be seen to concern surrogate entities as well. The notion of interpretation between theories (see e.g. Visser, 1997) gives rise to a suitable restrictive criterion for secondarily pure formal proofs, where each proof branch must begin with the derivation of an interpreted pure axiom. This restricts a proof to expressions referring to surrogate content only. To be even more accommodating in the particular formal proofs we consider pure, we can in some cases simplify the restricting criterion. We thus suggest that formal purity results can be exchanged between disciplines of mathematics, provided that

each theory draws upon its representation of pure entities only.

In short, we form a first understanding of how to consider purity for formal proofs. While the problem of finding a fully mechanical method that determines intuitive purity results remains open, we point out that informal conceptions can fruitfully go together with formal tools, in order to refine intuitions as well as bridge the gap between informal and formal concepts.

References

- Arana, Andrew (2009). “On formally measuring and eliminating extraneous notions in proofs”. In: *Philosophia Mathematica* 17.2, pp. 189–207.
- (2017). “On the alleged simplicity of impure proof”. In: *Simplicity: ideals of practice in mathematics and the arts*. Springer, pp. 205–226.
- Arana, Andrew and Michael Detlefsen (2011). “Purity of methods”. In: *Philosophers’ Imprint* 11.2, pp. 1–20.
- Baldwin, John T (2013). “Formalization, Primitive Concepts, and purity.” In: *Rev. Symb. Log.* 6.1, pp. 87–128.
- Detlefsen, Michael (2008). “Purity as an ideal of proof”. In: *The philosophy of mathematical practice*.
- Iemhoff, Rosalie (2017). “Remarks on simple proofs”. In: *Simplicity: ideals of practice in mathematics and the arts*. Springer, pp. 143–151.
- Kahle, Reinhard and Gabriele Pulcini (2017). “Towards an operational view of purity”. In: *The logica yearbook*.
- Lange, Marc (2019). “Ground and explanation in mathematics”. In: *Philosophers’ Imprint* 19.
- Lehet, Ellen (2021). “Impurity in Contemporary Mathematics”. In: *Notre Dame Journal of Formal Logic* 62.1, pp. 67–82.
- Visser, Albert (1997). “An overview of interpretability logic”. In: *Logic Group Preprint Series* 174.

Carnap's Problem for Generalised Quantifiers

Sebastian G.W. Speitel (University of Bonn)

Carnap's Problem concerns the question of how much of the semantics of an expression one can 'read off' of its inferential behaviour. More precisely, it asks what model-theoretic value of an expression is determined by a given consequence relation in the context of a particular semantic framework. Carnap [4] showed that even at the level of the propositional connectives the standard (single-conclusion) consequence relation of classical propositional logic is incapable of determining their standard truth-conditional semantics in any but the simplest cases. The underdetermination of the model-theoretic value of logical constants by consequence relations extends and deepens at the level of quantification. Recently, Bonnay & Westerståhl [3] characterised the extent to which the standard universal and existential quantifiers are underdetermined by the consequence relation of classical first-order logic (FOL). Their treatment of these expressions as quantifiers in the sense of generalised quantifier theory invites an extension of the investigation of the determination and underdetermination of quantifiers in the context of first-order consequence relations in general. To map out the framework for such an investigation, and to present some initial results, is the purpose of this talk.

A *generalised* or *Lindström-quantifier* \mathcal{Q} is a class of structures of the same signature [6]. Every quantifier \mathcal{Q} determines a unique *quantifier-on-a-domain* $M, \mathcal{Q}(M)$. Where \mathcal{L} is the language of FOL, we designate by $\mathcal{L}(\mathcal{Q})$ the language of FOL extended by a quantifier symbol \mathcal{Q} , and by $\mathcal{L}(Q_1, \dots, Q_n)$ the language of FOL extended by quantifier symbols Q_1, \dots, Q_n . Given the standard interpretations of the logical constants of FOL and an interpretation of \mathcal{Q} (interpretations of Q_1, \dots, Q_n) of appropriate type(s) we denote the model-theoretic consequence relation over the relevant logics by $\models_{\mathcal{Q}}$ ($\models_{\mathcal{Q}_1, \dots, \mathcal{Q}_n}$). An interpretation \mathcal{Q}' is *consistent with a consequence relation* $\models_{\mathcal{Q}}$ if $\models_{\mathcal{Q}} \subseteq \models_{\mathcal{Q}'}$. We then ask the following question:

for what values \mathcal{Q} , and under what conditions, is it the case that \mathcal{Q} is the unique interpretation of \mathcal{Q} that is consistent with $\models_{\mathcal{Q}}$?

In other words: under what conditions is the consequence relation $\models_{\mathcal{Q}}$ 'strong enough' to uniquely 'pin down' or determine the intended interpretation \mathcal{Q} of \mathcal{Q} ? When a quantifier is such that it is the unique \mathcal{Q} (satisfying certain conditions) consistent with $\models_{\mathcal{Q}}$, we say that it is *uniquely determined by* $\models_{\mathcal{Q}}$ (with respect to these conditions).

Bonnay and Westerståhl showed [3] that the demand that quantifiers be *isomorphism-invariant* suffices to uniquely determine the standard interpretation of the universal (and thus also the existential) quantifier in the context of the standard consequence relation of FOL. We show that the condition of isomorphism-invariance also renders several non-first-order definable quantifiers unique with respect to their associated consequence relation $\models_{\mathcal{Q}}$. In the class of type (1) cardinality quantifiers this is the case for, e.g.,

- (i) $\mathcal{Q} = \mathcal{Q}_0$, where $\mathcal{Q}_0(M) = \{A \subseteq M \mid \omega \leq |A|\}$ is the quantifier *there exist infinitely many*
- (ii) $\mathcal{Q} = \mathcal{Q}_{fin}$, where $\mathcal{Q}_{fin}(M) = \{A \subseteq M \mid |A| < \omega\}$ is the quantifier *there exist finitely many*

These results are, despite their elementary nature, philosophically interesting: both uniqueness and isomorphism-invariance have, in different traditions, been considered essential components of the *logicality* of an expression.¹ Unique determination of model-theoretic value by inference, in the sense outlined above, and under the assumption of further semantic constraints, thus delineates a class of logical constants far extending the usual class of first-order logical expressions. A criterion of logicality based on unique determination by inference (*categoricity*) and isomorphism-invariance (*formality*) was formulated and defended in [2].

In the class of type $\langle 1 \rangle$ cardinality quantifiers the ability to be uniquely determined by a consequence relation over a language of the form $\mathcal{L}(Q)$ appears to abruptly stop at \aleph_1 . Based on old results by Keisler and others² we show that the quantifier *there exist uncountably many* (Q_1), given by $Q_1(M) = \{A \subseteq M \mid \aleph_1 \leq |A|\}$, fails to be uniquely determinable over any consequence relation of the form $\models_{\mathcal{Q}}$. This result generalises, in a strengthened formulation, to various other classes of cardinality quantifiers of the form $Q_\alpha(M) = \{A \subseteq M \mid \aleph_\alpha \leq |A|\}$. If there is time, we will present further examples of quantifiers of various types that are not uniquely determined by their associated consequence relations.

Returning to a more abstract perspective and taking isomorphism-invariance to be a desirable constraint on potential interpretations of quantifier-expressions we further study the extent and limits of unique determinability of generalised quantifiers by appropriate consequence relations. In particular, we show that the EC_Δ -definability of a class in FOL is sufficient for a quantifier Q identified with that class to be uniquely determined by the consequence relation $\models_{\mathcal{Q}}$. We explore further relationships between definability and unique determination by a consequence relation and investigate several closure conditions of unique determination.

We conclude this talk by briefly looking at analogous questions for the case in which more than one generalised quantifier-expression is present in the language, i.e., at unique determinability with respect to consequence relations $\models_{Q_1, \dots, Q_n}$ over languages $\mathcal{L}(Q_1, \dots, Q_n)$, advancing some questions and conjectures, and reflecting on the philosophical significance of the results presented.

This talk is based on joint work with D. Bonnay and D. Westerståhl.

References

- [1] N. Belnap, “Tonk, Plonk and Plink.” *Analysis* 22, 1962, pp. 130-134.
- [2] D. Bonnay and S.G.W. Speitel, “The Ways of Logicality: Invariance and Categoricity”, in: *The Semantic Conception of Logic: Essays on Consequence, Invariance, and Meaning*, G. Sagi & J. Woods (eds.), Cambridge University Press 2021, pp. 55-80.
- [3] D. Bonnay and D. Westerståhl, “Compositionality Solves Carnap’s Problem”, *Erkenntnis* 81, 2016, pp. 721-739.
- [4] R. Carnap, *Formalization of Logic*, Harvard University Press 1943.
- [5] H.J. Keisler, “Logic with the Quantifier ‘There Exist Uncountably Many’”, *Annals of Mathematical Logic* 1, 1970, pp. 1-93.
- [6] P. Lindström, “First-Order Predicate Logic with Generalized Quantifiers”, *Theoria* 32, 1966, pp. 186-195.
- [7] G. Sher, *The Bounds of Logic: A Generalized Viewpoint*, MIT Press, 1991.

¹E.g. [1, 7].

²See [5]

Extracting Rules from Neural Networks with Partial Interpretations

Cosimo Persia, Ana Ozaki

University of Bergen, Norway

1 Introduction

Neural networks have been used to achieve important milestones in artificial intelligence [4, 10, 12, 8], but it is difficult to understand how predictions of the models are made, and this limits their usability. In this work previously published [16], we propose an approach for extracting rules from black-box machine learning models, such as neural networks. It is often the case that not all values in a dataset are known or trustable. For this reason, our approach assumes settings in which the dataset used to train the neural network contains missing values.

We first binarize a given dataset and we train a neural network with it. Then, we run the LRN algorithm [9]. This algorithm poses queries to the neural network, seen as a teacher, in order to extract rules encoded in it. Rules are represented using Horn logic, for example, they can be of the form $((\text{horse} \wedge \text{wings}) \rightarrow \text{pegasus})$. With Horn rules, we can carry automated reasoning in polynomial time, and it is feasible to check the quality of the model.

Related Work. Similar works [21, 15] extract probabilistic automata from neural networks by asking queries, and focus on how to better simulate queries asked to black-box models.

2 Extracting Rules

In this work, a neural network N can be seen as an alternative way of representing a formula t_N in propositional logic, that is a boolean function that receives a vector in $\{0, 1, ?\}^n$ and outputs a classification of this input in $\{0, 1\}$. The symbol ‘?’ stands for an ‘unknown value’ and n is the number of variables in the language.

The LRN algorithm learns an unknown target HORN formula t by posing queries to two kinds of oracles [9]. A *membership oracle* MQ_t is a function that takes as input a partial interpretation $\mathcal{I} \in \{0, 1, ?\}^n$ and it outputs ‘yes’ if \mathcal{I} satisfies t , ‘no’ otherwise. An *equivalence oracle* EQ_t takes as input a propositional formula h and it outputs ‘yes’ if $h \equiv t$, otherwise, it outputs a counterexample for t and h . A *membership query* is a call to MQ_t and an *equivalence query* is a call to EQ_t .

To simulate the membership oracle MQ_{t_N} , we directly use the classifier N . Whenever the algorithm calls MQ_{t_N} with input a partial interpretation \mathcal{I} , we check if $N(\mathcal{I}) = 1$. If so, we return the answer ‘yes’ to the algorithm, ‘no’ otherwise. We simulate EQ_{t_N} by generating a set of examples randomly and classifying the examples using membership queries. Then, we can search for examples in this set that the hypothesis constructed by LRN misclassifies. If the size of the set of examples generated randomly is at least $\frac{1}{\epsilon} \log_2 \left(\frac{|\mathcal{H}|}{\delta} \right)$ [20], with $|\mathcal{H}|$ being the size of the hypothesis space, then one can ensure that the hypothesis constructed is probably approximately

correct [19]. The parameter $\epsilon \in (0,1)$ indicates the probability that the hypothesis misclassifies an interpretation w.r.t. the target and $\delta \in (0,1)$ is the probability that the learned hypothesis errs more than ϵ . If the hypothesis space corresponds to the class of formulas only expressible with Horn logic and n variables, then the number of logically different hypothesis is close to $2^{\lfloor n/2 \rfloor}$ [1, 3]. This number follows from the fact that Horn logic is closed under intersection: if \mathcal{I} and \mathcal{I}' satisfy a Horn theory then $\mathcal{I} \cap \mathcal{I}'$ also does [11].

3 Experiments

We implemented the algorithm in a Python 3.9 script and we used the SymPy library [13] to express rules and check for satisfiability of formulas. For the neural networks, we used the Keras library [5]. Our LRN implementation can start with an empty hypothesis or with a set of Horn formulas as background knowledge (assumed to be true properties of the domain at hand). The background knowledge can also be used to check if the neural network model respects some desirable properties. We conduct the experiments on an Ubuntu 18.04.5 LTS with i9-7900X CPU at 3.30GHz with 32 logical cores, 32GB RAM.

We experiment our approach of extracting Horn theories from partial interpretations on a dataset in the medical domain, the hepatocellular carcinoma dataset [17]. This dataset contains missing values for attributes. We can consider each instance as a partial interpretation that sets some variables to true, some to false, and other variables to “unknown”.

Test Setting. In our experiments, we run the LRN algorithm and we set a limit of 100 equivalence queries that the algorithm can ask before terminating with the built hypothesis as its output. We compare the quality of the LRN hypothesis with the hypothesis formed by an incremental decision tree [7], an established white box machine learning model. We use “Hoeffding Decision Tree” implementation present in the “skmultiflow” framework [14]. The sampling idea for finding negative counterexamples for LRN is also used for extracting a decision tree from the neural network. When a counterexample is found, we incrementally train the tree with the entire sample.

Results. When the HORN hypothesis and the tree have been extracted, we compute truth tables and compare classifications. The percentage of interpretations that are labelled differently between the target and the hypothesis 9.2%, the target and the neural network 6.0%, the hypothesis and the neural network 5.8%, and the tree and the target 8.4%. The running time of the LRN algorithm with at most 100 equivalence queries was around 60 hours. The time for extracting an incremental decision tree is twice, around 120 hours. The type of rules that the LRN algorithm extracted are of the form:

$$\{\text{medium_hemoglobin_level} \wedge \dots \wedge \text{not_obese} \rightarrow \text{survives}\}$$

with around 40 different variables in the antecedent. With 100 equivalence queries, the hypothesis extracted has 20 rules of this type that are also present in the target t . Other rules that are logically entailed by t can be found in the hypothesis. Examples labelled negatively with many missing values contain more information about the dependency between variables that must be respected. Indeed, we noticed an increase of the accuracy of the neural network trained on more missing values ensuring ensured balanced classes. As a consequence, also the quality of the extracted rules improves.

References

- [1] V. Alekseev. On the number of intersection semilattices [in russian]. *DiskretnayaMat.1*, page 129–136, 1989.
- [2] D. Angluin. Queries and concept learning. *Machine Learning*, 2(4):319–342, 1988.
- [3] G. Burosch, J. Demetrovics, G. Katona, D. Kleitman, and A. Sapozhenko. On the number of closure operations. pages 91–105. János Bolyai Mathematical Society, Budapest, 1993.
- [4] M. Campbell, A. J. H. Jr., and F. Hsu. Deep blue. *Artif. Intell.*, 134(1-2):57–83, 2002.
- [5] F. Chollet et al. Keras, 2015.
- [6] L. De Raedt. Logical settings for concept-learning. *Artificial Intelligence*, 95(1):187–201, 1997.
- [7] P. M. Domingos and G. Hulten. Mining high-speed data streams. In *KDD*, pages 71–80. ACM, 2000.
- [8] D. Ferrucci. Introduction to “this is watson”. *IBM Journal of Research and Development*, 56:1:1–1:15, 05 2012.
- [9] M. Frazier and L. Pitt. Learning from entailment: An application to propositional horn sentences. In *ICML*, 1993.
- [10] S. Hölldobler, S. Möhle, and A. Tiginova. Lessons learned from alphago. In *YSIP2*, pages 92–101. CEUR-WS.org, 2017.
- [11] A. Horn. On sentences which are true of direct unions of algebras. *The Journal of Symbolic Logic*, 16(1):14–21, 1951.
- [12] Q. V. Le, M. Ranzato, R. Monga, M. Devin, G. Corrado, K. Chen, J. Dean, and A. Y. Ng. Building high-level features using large scale unsupervised learning. In *ICML*. icml.cc / Omnipress, 2012.
- [13] A. Meurer, C. Smith, M. Paprocki, O. Certík, S. B. Kirpichev, M. Rocklin, A. Kumar, S. Ivanov, J. K. Moore, S. Singh, T. Rathnayake, S. Vig, B. E. Granger, R. P. Muller, F. Bonazzi, H. Gupta, S. Vats, F. Johansson, F. Pedregosa, M. J. Curry, A. Terrel, S. Roucka, A. Saboo, I. Fernando, S. Kulal, R. Cimrman, and A. M. Scopatz. Sympy: symbolic computing in python. *PeerJ Comput. Sci.*, 3:e103, 2017.
- [14] J. Montiel, J. Read, A. Bifet, and T. Abdessalem. Scikit-multiflow: A multi-output streaming framework. *Journal of Machine Learning Research*, 19(72):1–5, 2018.
- [15] T. Okudono, M. Waga, T. Sekiyama, and I. Hasuo. Weighted automata extraction from recurrent neural networks via regression on state spaces. In *AAAI*, pages 5306–5314. AAAI Press, 2020.
- [16] C. Persia and A. Ozaki. Extracting rules from neural networks with partial interpretations. *Northern Lights Deep Learning Conference*, 2022.

- [17] M. S. Santos, P. H. Abreu, P. J. García-Laencina, A. Simão, and A. Carvalho. A new cluster-based oversampling method for improving survival prediction of hepatocellular carcinoma patients. *Journal of Biomedical Informatics*, 58:49–59, 2015.
- [18] A. Shih, A. Darwiche, and A. Choi. Verifying binarized neural networks by angluin-style learning. In *SAT*, 2019.
- [19] L. G. Valiant. A theory of the learnable. *Commun. ACM*, 27(11):1134–1142, 1984.
- [20] V. Vapnik. Chervonenkis: On the uniform convergence of relative frequencies of events to their probabilities. 1971.
- [21] G. Weiss, Y. Goldberg, and E. Yahav. Extracting automata from recurrent neural networks using queries and counterexamples. In *ICML*, volume 80, pages 5244–5253. PMLR, 2018.
- [22] Y. Zhang, P. Tiño, A. Leonardis, and K. Tang. A survey on neural network interpretability. *IEEE Transactions on Emerging Topics in Computational Intelligence*, 5(5):726–742, 2021.

Almost negative truth and fixpoints in intuitionistic logic

Mattias Granberg Olsson, University of Gothenburg
Graham Leigh, University of Gothenburg

We present work in progress on the relationship between the theory of transfinitely iterated strictly positive fixpoints and axiomatic theories of compositional and disquotational truth for almost negative formulae in intuitionistic logic. The starting point is the result of Cantini [1] and Feferman [2] (extended to the transfinite by Fujimoto [3]) that the (classical) theory of positive fixpoints $\widehat{\text{ID}}_1$, the Kripke-Feferman theory of compositional partial truth KF, and the uniformly disquotational theory for truth-positive formulae PUTB are mutually interpretable. We obtain similar results for the theories of transfinite iterations of strictly positive fixpoints (as in [4]) for almost negative operators, and disquotation for almost negative strictly truth-positive sentences, in intuitionistic logic ($\widehat{\text{ID}}_\alpha^i(\Lambda)$ and PAUTB_α^i respectively):

- First, $\widehat{\text{ID}}_\alpha^i(\Lambda)$ is interpretable in PAUTB_α^i , essentially by mimicking the classical proof.
- Second, PAUTB_α^i is interpretable (via a compositional theory) in $\widehat{\text{ID}}_{\omega \cdot \alpha}^i$ for limit α . This is achieved by using the extra ‘spacing’ between the levels, given by the multiplication by ω , to keep track of the nestings of implications in formulae.

References

- [1] Andrea Cantini. Notes on formal theories of truth. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 35(2):97–130, 1989.
- [2] Solomon Feferman. Reflecting on incompleteness. *The Journal of Symbolic Logic*, 56(1):1–49, March 1991.
- [3] Kentaro Fujimoto. Autonomous progression and transfinite iteration of self-applicable truth. *The Journal of Symbolic Logic*, 76(3):914–945, September 2011.
- [4] Christian Rüede and Thomas Strahm. Intuitionistic fixed point theories for strictly positive operators. *Mathematical Logic Quarterly*, 48(2):195–202, 2002.

Approximating trees as coloured linear orders and complete axiomatisations of some classes of trees*

Valentin Goranko¹ and Ruaan Kellerman²

¹ Stockholm University, Sweden, valentin.goranko@philosophy.su.se

² University of Pretoria, South Africa, ruaan.kellerman@up.ac.za

The logic-based study of linear orders was comprehensively presented in the early 1980s in the still very relevant classic book [10], and there have been several important further developments since then, mentioned below. In particular, the first-order (FO) theories of various naturally arising classes of linear orders are now well-known, for example, the FO theories of each ordinal α for $\alpha < \omega^\omega$ (see [10]), as well as the FO theory of the rational numbers (implicit in [10]). Furthermore, in [1] and [2] Doets studies several natural classes of *coloured* linear orders (i.e., linear orders enriched with unary predicates) and obtains complete axiomatisations for the first-order theories of: the class of coloured scattered linear orders; the class of coloured expansions of the natural numbers, from which the case of the integers follows easily; the class of coloured finite linear orders; the class of coloured complete linear orders; the class of coloured well-orders; and the class of coloured expansions of the order of the real numbers.

The study and axiomatisation of the first-order theories of naturally arising classes of trees is substantially more complicated, however, even when the first-order theory of the corresponding class of linear orders is known. A more systematic attempt was made in [4] to explore the general problem of transferring the first-order theory of a class of linear orders to the class of trees whose paths are all contained in that class of linear orders. That work left many open questions and indicated some inherent difficulties. They are mainly due to the following facts:

- (i) Since paths (maximal linearly ordered chains) are special sets of nodes in a given tree, the first-order language for trees cannot, in general, impose first-order properties on all paths of the tree, but only on the *first-order definable* ones. A path P in a tree $\mathfrak{T} := (T; <)$ is called *singular* when it contains a node u such that the set $\{x \in T : u \leq x\}$ is a linear order within \mathfrak{T} . All singular paths are parametrically definable. However, trees may also contain *emerging* paths, which are paths that are not singular. In such non-definable emerging paths, behaviour in the terminal part of the path cannot generally be controlled within the expressive means of first-order logic.
- (ii) The branching structure of a tree cannot be captured by the properties of its paths.

Consequently, there are very few known complete axiomatisations of first-order theories of classes of trees, in essence comprising the following classes: the class of finitely branching trees (implicit in [11]), the class of (ordinary or coloured) well-founded trees (see [2]), and the class of finite trees (see [9]). Also, [3] contains some general results on axiomatising subclasses of the class of finitely branching trees relative to the respective classes of trees with no restriction on their branching. Further, the first-order theories of the class of trees, all of whose paths contain greatest elements (leaves), and the class of trees whose paths are all isomorphic to some given

*This talk is based on the paper [8].

ordinal α with $\alpha < \omega^\omega$, are investigated in [7], but without deriving complete axiomatisations of these first-order theories. Lastly, even though not directly related to the present work, we should mention the very important works by Gurevich and Shelah [6] and [5] on decidability of first-order theories of coloured trees with additional quantification over branches.

The goal of the present work is to study and axiomatise in first-order logic the classes of trees naturally arising from some important linear orders. More precisely, we obtain axiomatisations of the first-order theories of these classes of trees, rather than axiomatisations of those classes themselves. This amounts to the following: given a class of trees \mathcal{K} , we seek a recursive (i.e., decidable) set of first-order sentences Σ such that $\Sigma \subseteq \text{Th}(\mathcal{K})$ and $\Sigma \models \text{Th}(\mathcal{K})$. In turn, $\Sigma \models \text{Th}(\mathcal{K})$ if and only if for each natural number n and each model \mathfrak{T} of Σ , there exists a tree \mathfrak{S} in \mathcal{K} such that \mathfrak{T} and \mathfrak{S} satisfy the same sentences of quantifier rank at most n .

Now for any order type α , a tree whose paths are all isomorphic to α is called an α -tree. This work addresses and solves the problems of axiomatising the first-order theories of the classes of (coloured) ω -trees, ζ -trees, η -trees and λ -trees, where ω , ζ , η and λ are the order types of the sets of natural numbers, integers, rational numbers, and real numbers, respectively. While the case of η -trees is easy and the case of ω -trees was essentially known from [2], the cases of ζ -trees and λ -trees turned out to be quite non-trivial. The complete axiomatisations of their first-order theories are obtained here by using a new construction for approximating a given tree by a suitably coloured linear order and then using the axiomatisations of the first-order theories of the classes of coloured expansions of ζ and λ respectively.

References

- [1] K. Doets. *Completeness and Definability: Applications of the Ehrenfeucht Game in Second-Order and Intensional Logic*. PhD thesis, University of Amsterdam, 1987.
- [2] K. Doets. Monadic Π_1^1 -theories of Π_1^1 -properties. *Notre Dame Journal of Formal Logic*, 30(2):224–240, 1989.
- [3] V. Goranko. Trees and finite branching. In *Proceedings of the 2nd Panhellenic Logic Symposium*, pages 91–101, 1999.
- [4] V. Goranko and R. Kellerman. Classes and theories of trees associated with a class of linear orders. *Logic Journal of the IGPL*, 19:217–232, 2010.
- [5] Y. Gurevich and S. Shelah. The decision problem for branching time logic. *The Journal of Symbolic Logic*, 50:668–681, 1985.
- [6] Y. Gurevich and S. Shelah. To the decision problem for branching time logic. In G. Dorn and editors P. Weingartner, editors, *Foundations of Logic and Linguistics: Problems and Their Solutions*, pages 181–198. Plenum, 1985.
- [7] R. Kellerman. First-order theories of bounded trees, 2019. under submission.
- [8] R. Kellerman and V. Goranko. Approximating trees as coloured linear orders and complete axiomatisations of some classes of trees. *J. Symb. Log.*, 86(3):1035–1065, 2021.
- [9] J. Rogers R. Backofen and K. Vijay-Shankar. A first-order axiomatization of the theory of finite trees. *Journal of Logic, Language and Information*, 4:5–39, 1995.
- [10] J.G. Rosenstein. *Linear Orderings*. Academic Press, 1982.
- [11] J. Schmerl. On \aleph_0 -categoricity and the theory of trees. *Fundamenta Mathematicae*, 94:121–128, 1977.

Proof-theoretic Semantics in Sheaves (Extended Abstract)

David Pym Eike Ritter Edmund Robinson
University College London University of Birmingham Queen Mary University of London

In proof-theoretic semantics [6], model-theoretic validity is replaced by proof-theoretic validity. Validity of formulae is defined inductively from a base giving the validity of atoms using inductive clauses derived from proof-theoretic rules. A key aim is to show completeness of the proof rules without any requirement for formal models. Establishing this for propositional intuitionistic logic (IPL) raises some technical and conceptual issues [2, 3, 5].

We relate the (complete) base-extension semantics of [5] to categorical proof theory and sheaf-theoretic semantics (e.g., [1]). For the latter, propositions are interpreted as functors from a category of bases to the lattice $\{\{\top\}, \emptyset\}$. This set of functors forms the truth values of a topos of functors from bases to **Set**. There are two critical aspects: the stability of interpretation under extension of bases lands us in the world of Kripke models, and the non-standard interpretation of disjunction is revealed to come from a Grothendieck topology.

Base-extension Semantics in Presheaves. Sandqvist [5] gives a base-extension proof-theoretic semantics for IPL for which natural deduction is sound and complete. A base \mathcal{B} is a set of atomic rules (for $\vdash_{\mathcal{B}}$) as in Definition 1, which also defines the application of base rules, and satisfaction in a base ($\Vdash_{\mathcal{B}}$). Roman p, P , etc. denote atoms and sets of atoms; Greek ϕ, Γ , etc. denote formulae and sets of formulae.

Definition 1 (Sandqvist’s Semantics) *Base rules \mathcal{R} , application of base rules, and satisfaction of formulae in a (possibly finite) countable base \mathcal{B} of rules \mathcal{R} are defined as follows:*

$$\begin{array}{c}
 \frac{[P_1] \quad \dots \quad [P_n]}{r} \mathcal{R} \quad \text{(Ref) } P, p \vdash_{\mathcal{B}} p \\
 \text{(App}_{\mathcal{R}}) \text{ if } ((P_1 \Rightarrow q_1), \dots, (P_n \Rightarrow q_n)) \Rightarrow r \text{ and, for all } i \in [1, n], \\
 P, P_i \vdash_{\mathcal{B}} q_i, \text{ then } P \vdash_{\mathcal{B}} r
 \end{array}$$

$$\begin{array}{ll}
 \text{(At) for atomic } p, \Vdash_{\mathcal{B}} p \text{ iff } \vdash_{\mathcal{B}} p & \text{(}\vee\text{) } \Vdash_{\mathcal{B}} \phi \vee \psi \text{ iff, for every atomic } p \text{ and every } \mathcal{C} \supseteq \mathcal{B}, \\
 & \text{if } \phi \Vdash_{\mathcal{C}} p \text{ and } \psi \Vdash_{\mathcal{C}} p, \text{ then } \Vdash_{\mathcal{C}} p \\
 \text{(}\supset\text{) } \Vdash_{\mathcal{B}} \phi \supset \psi \text{ iff } \phi \Vdash_{\mathcal{B}} \psi & \text{(}\perp\text{) } \Vdash_{\mathcal{B}} \perp \text{ iff, for all atomic } p, \Vdash_{\mathcal{B}} p \\
 \text{(}\wedge\text{) } \Vdash_{\mathcal{B}} \phi \wedge \psi \text{ iff } \Vdash_{\mathcal{B}} \phi \text{ and } \Vdash_{\mathcal{B}} \psi & \text{(Inf) for } \Theta \neq \emptyset, \Theta \Vdash_{\mathcal{B}} \phi \text{ iff, for every } \mathcal{C} \supseteq \mathcal{B}, \text{ if } \Vdash_{\mathcal{C}} \theta \\
 & \text{for every } \theta \in \Theta, \text{ then } \Vdash_{\mathcal{C}} \phi
 \end{array}$$

There is a substitution (cut) operation on bases that maps derivations $P \vdash_{\mathcal{B}} p$ and $p, Q \vdash_{\mathcal{B}} q$ to a derivation $P, Q \vdash_{\mathcal{B}} q$.

Key to understanding our categorical formulation is the Yoneda lemma (see [1]): let \mathcal{C} be a locally small category, let **Set** be the category of sets, and $F \in [\mathcal{C}^{op}, \mathbf{Set}]$ (the category of presheaves over \mathcal{C}); then, for each object C of \mathcal{C} , with $h^C = \text{hom}(-, C)$, the natural transformations $\text{Nat}(h^C, F) \equiv \text{hom}(\text{hom}(-, C), F) \cong F(C)$.

We give a category-theoretic formulation of proof-theoretic validity using presheaves (i.e., functors $F \in [\mathcal{W}^{op}, \mathbf{Set}]$), where \mathcal{W} has objects pairs (\mathcal{B}, P) and morphisms are given by conclusions of the base and derivations in the larger base. Composition is given by substitution.

Define a functor $\llbracket \phi \rrbracket: \mathcal{W}^{op} \rightarrow \mathbf{Set}$ by induction over the structure of ϕ as follows: the base case $\llbracket p \rrbracket(\mathcal{B}, P)$ is the set of derivations $P \vdash_{\mathcal{B}} p$. $\llbracket p \rrbracket$ applied to morphisms is given by substitution. The definition is extended to the connectives homomorphically. A key step is the use of the Yoneda lemma to define the (hom-set) interpretation of \supset , which is used to define the interpretation of Sandqvist's (elimination-style) semantics for \vee (see also below). Thus we establish the formal functoriality and naturality of Sandqvist's semantics.

Theorem 2 (Soundness & Completeness) *Define (cf. [5]) $\Gamma \Vdash \phi$ as: for all \mathcal{B} , if $\Vdash_{\mathcal{B}} \psi$ for all $\psi \in \Gamma$, then $\Vdash_{\mathcal{B}} \phi$. Then $\Gamma \vdash \phi$ (in natural deduction for IPL, cf. [5]) iff $\Gamma \Vdash \phi$.*

The proof of soundness uses the existence of a natural transformation corresponding to \Vdash : $\Gamma \Vdash \phi$ iff there exists a natural transformation from $\llbracket \Gamma \rrbracket$ to $\llbracket \phi \rrbracket$. The proof of completeness uses a special base, as in [5], which is extended via $\llbracket - \rrbracket$ to the full consequence relation.

Sheaves and Disjunction. Standard Kripke semantics interprets both conjunction and disjunction pointwise (i.e., on each base, in proof-theoretic semantics [3]), while it relies on the extension ordering for implication (cf. the discussion of Goldfarb's semantics in [3]). This is a result of the requirement that the set of bases validating any proposition should be closed under extension: propositions do not become untrue if we are given additional atomic information. But there is an issue over the interpretation of disjunction. A standard constructive view is that the proof of a disjunction should resolve to a proof of one of the disjuncts. This is not obviously stable under extension of information and obtaining a pointwise disjunction reflecting this viewpoint is the hardest part of the proof of completeness of standard Kripke models for IPL. We show that Sandqvist's approach avoids this difficulty by using a Grothendieck topology.

In this section, we ignore differences between derivations, and interpret propositions as truth values in the topos $\mathcal{S} = [\mathcal{W}^{op}, \mathbf{Set}]$. These can be identified with subfunctors of the constant singleton functor $\{\top\}$ (cf. [1]). Atomic propositions are interpreted in \mathcal{S} via $\llbracket p \rrbracket(\mathcal{B}, P) = \{\top \mid P \vdash_{\mathcal{B}} p\} = \{\top \mid P \Vdash_{\mathcal{B}} p\}$. Sandqvist's satisfaction conditions for conjunction and implication correspond to the internal interpretation of the logic in the topos \mathcal{S} , but his conditions for disjunction and false do not.

For each atomic proposition p , we form an internal operator on truth values: $j_p(\omega) = (\omega \supset \llbracket p \rrbracket) \supset \llbracket p \rrbracket$. The set of atomic propositions internalizes as the constant functor: $At(\mathcal{B}) = \{p \mid p \text{ is atomic}\}$. Consider the function on truth values that is the internal interpretation of $j(\omega) = \forall p \in At. j_p(\omega) = \forall p \in At. (\omega \supset \llbracket p \rrbracket) \supset \llbracket p \rrbracket$. This is a Lawvere-Tierney topology — that is, the internalization of a Grothendieck topology — and each $\llbracket p \rrbracket$ is j -closed.

Sandqvist's satisfaction conditions correspond exactly to the standard interpretation of connectives in the topos of sheaves for this topology.

Proposition 3 *For any proposition ϕ , and any world $W = (\mathcal{B}, P)$, $P \Vdash_{\mathcal{B}} \phi$ iff $\llbracket \phi \rrbracket(\mathcal{B}, P) = \{\top\}$, where $\llbracket \phi \rrbracket$ is the standard interpretation of ϕ in $Sh_j(\mathcal{S})$.*

This follows from the closure of sheaves under conjunction and implication, intuitionistic equivalence of $((\phi \vee \psi) \supset p) \supset p$ and $((\phi \supset p) \wedge (\psi \supset p)) \supset p$, and expansion of definitions.

This sheaf model can be seen as a continuation semantics in which a complete proof-search [4] is the proof of an atomic proposition. Using a topology for this results in a disjunction being valid iff a point is covered by refinements on each of which one of the disjuncts holds — cf. Beth’s semantics (see, e.g., [1]).

References

- [1] J. Lambek and P. Scott. *Introduction to higher-order categorical logic*. Cambridge studies in advanced mathematics 7, Cambridge University Press, 1986.
- [2] T. Piecha. Completeness in Proof-theoretic Semantics. In: *Advances in Proof-theoretic Semantics*, T. Piecha P. Schroeder-Heister (eds), Trends in Logic 43, 231-252, Springer 2016.
- [3] T. Piecha and P. Schroeder-Heister. Incompleteness of Intuitionistic Propositional Logic with Respect to Proof-Theoretic Semantics. *Studia Logica* 107, 233-246, 2019.
- [4] D. Pym and E. Ritter. *Reductive Logic and Proof-search: Proof Theory, Semantics, and Control*. OUP, 2004
- [5] T. Sandqvist. Base-extension semantics for intuitionistic sentential logic. *Logic Journal of the IGPL*, 23(5), 719-731, 2015.
- [6] P. Schroeder-Heister. Proof-theoretic versus Model-theoretic Consequence. In: *The Logica Yearbook 2007*. Prague: Filosofia 2008, 187-200.

Monotonicity under Knowledge

Jialiang Yan¹ and Maria Aloni²

¹ILLC, University of Amsterdam & Tsinghua University

²ILLC, University of Amsterdam

The paper discusses cases of monotonic reasoning under knowledge which appear to be somehow paradoxical. Consider (1):

- (1) a. Tom knows that Mary is a rich woman
- b. Tom knows that Mary is a woman

The monotonic inference from (1-a) to (1-b) is sound and valid given the classical semantics for *know*. But intuitively there is something strange that prevents us from accepting the reasoning as completely safe. In the process of deriving (1-b), there is loss of information, i.e., the characteristic of Mary’s wealth has been omitted. This loss of information does not seem to matter if we discuss what Tom knows in a neutral situation. But once the feature of Mary’s wealth becomes salient in context, the information loss might lead to problems in practice. Consider a scenario where Tom wants to marry a rich woman. Then based on (1-b), which is warranted by monotonicity, will you predict that Tom will marry Mary? (1-b) is compatible with a situation where Tom doesn’t know whether Mary is rich and hence, solely on the basis of (1-b), it is hard to judge whether he wants to marry her on this matter.

We argue that the cause of such decision-making problem is that some additional inferences are licensed by (1-b), which were not licensed by (1-a) despite the fact that the latter is more informative.

- (2) a. Tom knows that Mary is a woman
- b. \leadsto It is consistent with Tom’s knowledge that Mary is a rich woman, **and** it is consistent with Tom’s knowledge that she is a non-rich woman.

We then arrive at (2-b) from (1-a), which appears to be paradoxical:

Tom knows that Mary is a rich woman

↓ *Monotonicity*

Tom knows that Mary is a woman

↓ *Inference in (2)*

It is consistent with Tom’s knowledge that she is a non-rich woman

To address this paradox, we argue that the inference in (2) is derived as a case of epistemic Free Choice (FC, see [5, 6, 1], etc.) inference, which is triggered by a disjunctive reinterpretation of the predicate *woman*. To show this, we first assume that a predication can be reinterpreted as a disjunction in a given context, and so it conveys a disjunctive meaning. Evidence from the

Exemplar Theory w.r.t concept learning in cognitive psychology shows that concepts are typically represented as remembered (hence salient) instances (see [4, 7], etc.). So in conversations, when representing the meaning of a predicate people focus on a part of the objects in its extension made salient by the question under discussion. As a result we obtain a disjunctive representation of the predicate consisting of the union of the salient objects (first disjunct) and the remaining objects (second disjunct). The latter is normally denoted as the negation of the former.

In this way, the predicate *woman* can be reinterpreted as *rich woman or non-rich woman*, in a context where *rich woman* becomes salient. We call *rich woman* and *non-rich woman* **sub-predicates** of *woman*, because they are semantically included in *woman*. We further assume that a predicate can only be disjunctively reinterpreted by its sub-predicates.

Once we reinterpret the relevant predicate as a disjunction, the inference in (2) can be derived as a FC inference, and thus leads to the paradox:

- (3) a. Tom knows that Mary is a woman / $[K](W(m))$
 b. \Leftrightarrow Tom knows that Mary is a rich woman or a non-rich woman / $[K](R(m) \vee \neg R(m))$
 c. \rightsquigarrow It is consistent with Tom's knowledge that Mary is a not rich woman. / $\langle K \rangle \neg R(m)$

The inference from (3-a) to (3-b) is due to the disjunctive reinterpretation of *woman*. We then apply the FC principle in (4-b) to derive (3-c) from (3-b). The principle in (4-b) is so-called \Box -free choice, where a conjunctive meaning of possibility modals can be derived from a disjunctive statement under necessity:

- (4) a. You ought to send the letter or burn it \rightsquigarrow you are permitted to send the letter, and you are permitted to burn it.
 b. $\Box(p \vee q) \rightsquigarrow \Diamond p \wedge \Diamond q$

In fact, (3) provides a disjunction free example of a FC inference, by some reinterpreting process. To capture the inferences in (3), we develop a formal framework based on (Quantified) Bilateral State-based Modal Logic ((Q)BSML) proposed by Aloni ([2, 3]). We skip the description of the system, which can be found in [2, 3, 9], and only show the new definition of *reinterpretation function* in the following, which only applies in case saliency is satisfied.

Definition 1 (Reinterpretation function) *Let Nfo be the set of all NE-free formulas of the language, and Prt be the set of all atomic predications. A reinterpretation function $\|Nfo\|_{P\bar{x}}$ is a mapping from $Nfo \times Prt$ to NE-free formulas.*

- $\|Q\bar{x}\|_{P\bar{x}} = \begin{cases} P_Q\bar{x} \vee \neg P_Q\bar{x} & \text{if } P \subset Q \\ Q\bar{x} & \text{otherwise} \end{cases}$
- $\|\phi \vee \psi\|_{P\bar{x}} = \|\phi\|_{P\bar{x}} \vee \|\psi\|_{P\bar{x}}$
- $\|\neg\phi\|_{P\bar{x}} = \neg \|\phi\|_{P\bar{x}}$
- $\|\exists x\phi\|_{P\bar{x}} = \exists x \|\phi\|_{P\bar{x}}$
- $\|\phi \wedge \psi\|_{P\bar{x}} = \|\phi\|_{P\bar{x}} \wedge \|\psi\|_{P\bar{x}}$
- $\|[K]\phi\|_{P\bar{x}} = [K]\|\phi\|_{P\bar{x}}$

The operator $\langle K \rangle$, which stands for “it is consistent with someone’s knowledge”, can be defined as the dual of $[K]$. The function intuitively says that, a predicate Q can be reinterpreted syntactically as $\lambda x[P_Qx \wedge \neg P_Qx]$ (where P_Qx stands for $Px \wedge Qx$) if P is a sub-predicate of Q . In our example, we have the reinterpretation: $\|Woman\ x\|_{Rich\ x} = Rich_{Woman}\ x \vee \neg Rich_{Woman}\ x$.

One of the main consequences of (Q)BSML is that the \Box -free choice can be derived. As a result, using the reinterpretation function and the principle of \Box -free choice, the inferences in

(3) can be captured. Therefore a formally pragmatic account of the paradox has been provided.

In this formal framework, we will also capture epistemic contradiction (see[8]) and the contrast between the belief and knowledge with respect to their interaction with epistemic possibility *might*.

References

- [1] Maria Aloni. Free choice, modals, and imperatives. *Natural Language Semantics*, 15(1):65–94, 2007.
- [2] Maria Aloni. Logic and conversation: the case of free choice. Manuscript, ILLC, University of Amsterdam, 2022.
- [3] Maria Aloni and Peter van Ormondt. Modified numerals and split disjunction: the first-order case. Manuscript, ILLC, University of Amsterdam, 2021.
- [4] Lee R Brooks. Decentralized control of categorization: The role of prior processing episodes. 1987.
- [5] Danny Fox. Free choice and the theory of scalar implicatures. In *Presupposition and implicature in compositional semantics*, pages 71–120. Springer, 2007.
- [6] Hans Kamp. Free choice permission. In *Meaning and the Dynamics of Interpretation*, pages 169–184. Brill, 2013.
- [7] Jean-Pierre Thibaut and Sabine Gelaes. Exemplar effects in the context of a categorization rule: Featural and holistic influences. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, 32(6):1403, 2006.
- [8] Seth Yalcin. Epistemic modals. *Mind*, 116(464):983–1026, 2007.
- [9] Jialiang Yan and Maria Aloni. A puzzle about monotonicity under knowledge. Manuscript, ILLC, University of Amsterdam, 2022.

Generalizing Rules via Algebraic Constraints (Extended Abstract)

Alexander V. Gheorghiu

David J. Pym

University College London, London WC1E 6BT, United Kingdom

Logics are often characterized by proof systems that are composed of rules. These rules give meaning to the logic — see, for example, proof-theoretic semantics [10]. We [6] propose a framework called *generalizing rules via algebraic constraints* (GRvAC), within which a rule may be decomposed into another rule together with some constraints over an algebra. The effect on the logic as a whole is more easily understood in the other direction: one enriches a logic \mathcal{L} with an algebra \mathcal{A} to form a presentation of another logic \mathcal{L}' . In short, we make precise the meaning of equations of the following form:

$$\text{Proof in } \mathcal{L}' = \text{Proof in } \mathcal{L} + \text{Algebra of Constraints } \mathcal{A}$$

By doing reasoning in \mathcal{L} enriched by \mathcal{A} , one recovers reasoning in \mathcal{L}' through a transformation that is parameterized by solutions to the algebraic constraints. Consequently, \mathcal{L} is thought of as more *general* than \mathcal{L}' . More precisely, one begins by labling the syntax of \mathcal{L} by (a syntax for) \mathcal{A} so that assignments I of the variables of \mathcal{A} determine valuations ν_I mapping the syntax of \mathcal{L} enriched by \mathcal{A} to the syntax of \mathcal{L}' . A rule of \mathcal{L}' is *generalized* when a rule of \mathcal{L} (taken over the enriched language) with constraints (i.e., equations) over \mathcal{A} is used to express it.

As an example, consider the *resource-distribution via boolean constraints* (RDvBC) mechanism introduced by Harland and Pym [8], of which GRvAC framework is an abstraction. The RDvBC mechanism was introduced for the study of proof-search in the presence of multiplicative (or intensional) connectives, such as for proof-search in linear logic (LL). One labels the formulas of LL with a syntax for boolean algebra \mathcal{B} (e.g., one has formulas $\phi \cdot x, \psi \cdot \bar{x}$ in which ϕ and ψ are formulas of LL, x is a boolean variable, and \bar{x} is its negation) such that assignments I determine valuations ν_I that keep formulas labelled by variables that I map to 1 and delete formulas labelled by variables that I map to 0 (e.g., if $I(x) = 0$, then $\nu_I(\{\phi \cdot x, \psi \cdot \bar{x}\}) = \{\phi\}$). This setup allows multiplicative rules to be *generalized* to additive rules; for example,

$$\frac{\Gamma \vdash \phi \quad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi \otimes \psi} \quad \text{generalizes to} \quad \frac{\Gamma \cdot x, \Delta \cdot \bar{x} \vdash \phi \quad \Gamma \cdot x, \Delta \cdot \bar{x} \vdash \psi}{\Gamma, \Delta \vdash \phi \otimes \psi}$$

In terms of GRvAC, this witnesses the following equation in which LL is a proof system for linear logic and LK is a proof system for classical logic: $\text{LL} = \text{LK} + \mathcal{B}$. Other examples of GRvAC are present in the literature too; for example, algebraic constraints may be used for unification in logic programming, which can be understood as saying that propositional logic is more general than predicate logic — see [5] for details.

Though generalization allows one to relate two logics, the idea of algebraic constraints is useful in itself and present elsewhere in the literature — see, for example, work by Negri [9] on relational calculi. Indeed, the concept of enrichment here is strongly related to the framework of *Labelled Deductive Systems* introduced by Gabbay [4].

The GRvAC framework is useful both for the theory and practice of logic. In theory, it is a technology that allows one to express formally relationships *between* logics; for example, it supports the folklore that classical logic (CL) is a combinatorial core of logics (i.e., CL generalizes most logics). It also allows one to study metatheory for particular logics; for example, the GRvAC framework allows one to translate between nested systems, tableaux systems, and relational calculi for normal modal logics, thereby proving soundness and completeness of all by proving it for one [6] (i.e., if there is a proof witnessing a sequent in one system, then immediately there is a proof witnessing the sequent in the other systems).

By understanding how a logic arises from CL by means of an algebra, GRvAC allows one to derive model-theoretic semantics for the logic; and, conversely, it allows one to generate sound and complete proof systems for a logic from a frame semantics. The semantic uses of GRvAC are prefigured by Docherty [3]. An example of the effectiveness of the GRvAC framework for metatheory is captured by a case study on intuitionistic logic (IL). Here, GRvAC allow one to construct from a single-conclusioned calculus a multiple-conclusioned sequent calculus, which witnesses that CL is the combinatorial core of IL. By studying the new calculus' relationship to CL using GRvAC, one can *derive* a model-theoretic semantics of IL. The derivations provides a new technique for proving soundness and completeness that proceeds by showing the equivalence of proof-search of the two logics relative to the constraints captured by the algebra — see [7] for further discussion.

The practical uses of GRvAC are in proof-search (including algorithmic). This claim is justified by the examples above; that is, RDvBC concerns the context-management problem during proof-search in substructural logics, the multiple-conclusion system for IL is a powerful tool for doing proof-search with backtracking. In general, GRvAC allows one to separate the combinatorial aspects of a logic from the internal choices made during proof-search; that is, the combinatorial aspects of proof-search in \mathcal{L}' can be understood by proof-search in \mathcal{L} with controls governed by constraints over \mathcal{A} . Among other things, therefore, GRvAC allows one to capture certain amount of global reasoning during proof-search, which can be interpreted as capturing a certain amount of backtracking within a proof system.

To elucidate the usefulness of GRvAC in proof-search, we illustrate its application to quantifiers. This captures earlier work by Wallen [11], Andrews [1], and Bibel [2]. Consider the putative conclusion $\exists x \forall y Pxy \vdash \forall u \exists v Puv$ in classical first-order logic (FOL). Two proof-search attempts are as follows:

$$\frac{\frac{P(a, b) \vdash \forall u \exists v P(u, v)}{\forall y P(a, y) \vdash \forall u \exists v P(u, v)} \forall_L}{\exists x \forall y P(x, y) \vdash \forall u \exists v P(u, v)} \exists_L \qquad \frac{\frac{\frac{Pab \vdash Pab}{Pab \vdash \exists u Pvb} \exists_R}{\forall y Pay \vdash \exists u Pvb} \forall_L}{\forall y Pay \vdash \forall u \exists v Pvu} \forall_R}{\exists x \forall y Pxy \vdash \forall u \exists v Pvu} \exists_L$$

The first proof-search fails and the second succeeds. Why does the first fail? The GRvAC framework may be used to understand these proof-searches. One can generalize the quantifier rules so as not to commit to a substitution, but rather track that some substitution needs to be

made, together with its conditions; for example, one has a computation of the following form:

$$\begin{array}{c}
\frac{\mathfrak{A}_{A \in \Gamma} \mathfrak{A}_{B \in \Delta} \ell(A) = \ell(B)}{\Gamma \vdash \Delta} \\
\frac{\Gamma, \phi[x \mapsto x \cdot n] \vdash \Delta \quad \kappa}{\Gamma, \exists x \phi \vdash \Delta} \\
\frac{\Gamma \vdash \phi[x \mapsto x \cdot n], \Delta \quad \kappa}{\Gamma \vdash \forall x \phi, \Delta}
\end{array}
\qquad
\begin{array}{c}
\frac{n = a \quad m = b}{P(x \cdot n, b) \vdash P(a, u \cdot m)} \\
\frac{P(x \cdot n, b) \vdash \exists v P(v, u \cdot m) \quad m \neq n, b}{P(x \cdot n, b) \vdash \exists v P(v, u \cdot m)} \\
\frac{P(x \cdot n, b) \vdash \exists v P(v, u \cdot m)}{\forall y P(x \cdot n, y) \vdash \forall u \exists v P(v, u)} \\
\frac{\forall y P(x \cdot n, y) \vdash \forall u \exists v P(v, u)}{\exists x \forall y P(x, y) \vdash \forall u \exists v P(v, u)}
\end{array}$$

— the constraint κ expresses that n is not any term or label that appears in Γ, ϕ or Δ , the notation $\mathfrak{A}_{x \in X} \kappa_x$ denotes a meta-disjunction over constraints κ_x , for each $x \in X$, and $\ell(\phi)$ is a list of the labels occurring in ϕ .

The insolubility of the constraints $n = a$, $m \neq n, b$ and $m = b$ means that there is no interpretation of the proof structure as a proof. Nonetheless, the constraints give information about why the reduction fails that may be leveraged through some *global* reasoning to yield a successful proof-search. There is no purpose in permuting the rules producing m and n with each other as in either case one would have $m \neq n$, but the substitutions producing a and b are free of constraints, hence one can permute the rule producing b with the rule producing m , thereby eliminating the constraint $m \neq b$. The result is a coherent set of constraints whose solution determines the successful proof-search attempt.

The GRvAC framework allows one to express complex rules as simple rules together with algebraic constraints that recover the former from the latter by means of transformations parameterized by solutions to equations over the algebra. It is useful for intra-logic metatheory (i.e., proof theory and semantics), for inter-logic metatheory (i.e., connexions between logics), and in applied logic tasks involving proof-search. Though we have outlined it conceptionally, substantial work remains in developing the space of examples and using it to develop uniform approaches to metatheory. Moreover, on the question of proof-search, GRvAC may be used to give a general mathematical theory of control, which is currently lacking, and relate the control problems of proof-search to other aspects of the logic (e.g., the clauses of its semantics).

References

- [1] ANDREWS, P. B. Theorem Proving via General Matings. *Journal of the ACM* 28, 2 (1981), 193–214.
- [2] BIBEL, W. *Automated Theorem Proving*. Springer, 1982.
- [3] DOCHERTY, S., AND PYM, D. J. Modular Tableaux Calculi for Separation Theories. In *Foundations of Software Science and Computation Structures — FOSSACS (2018)*, C. Baier and U. Dal Lago, Eds., pp. 441–458.
- [4] GABBAY, D. M. *Labelled Deductive Systems*. Oxford University Press, 1996.
- [5] GHEORGHU, A. V., DOCHERTY, S., AND PYM, D. J. Reductive Logic, Coalgebra, and Proof-search: A Perspective from Resource Semantics. In *Samson Abramsky on Logic and Structure in Computer Science and Beyond*, A. Palmigiano and M. Sadrzadeh, Eds., Springer Outstanding Contributions to Logic Series. Springer, 2021. In press, 2022.
- [6] GHEORGHU, A. V., AND PYM, D. J. Generalizing Rules via Algebraic Constraints. <http://www0.cs.ucl.ac.uk/staff/D.Pym/grvac.pdf>, 2022. Submitted.
- [7] GHEORGHU, A. V., AND PYM, D. J. Semantics ex Proof and Refutation. <http://www0.cs.ucl.ac.uk/staff/D.Pym/semantics-ex-proof-ref.pdf>, 2022. Presented at *Bilateralism & Proof-Theoretic Semantics*, Ruhr University Bochum, March 17-18, 2022. <https://sites.google.com/view/conferencebilateralism2022/home>
- [8] HARLAND, J., AND PYM, D. J. Resource-distribution via Boolean Constraints. *ACM Transactions on Computational Logic* 4 (2003), 56–90.
- [9] NEGRI, S. Proof Analysis in Modal Logic. *Journal of Philosophical Logic* 34, 5 (2005), 507–544.
- [10] SCHROEDER-HEISTER, P. Proof-Theoretic Semantics. In *The Stanford Encyclopedia of Philosophy*, E. N. Zalta, Ed. Metaphysics Research Lab, Stanford University, 2018.
- [11] WALLEN, L. A. *Automated Proof Search in non-Classical Logics: Efficient Matrix Proof Methods for Modal and Intuitionistic Logics*. PhD thesis, The University of Edinburgh, 1987.

“NEW FOUNDATIONS” FOR METAPHYSICS

Violeta Conde and Alejandro Gracia Di Rienzo

Abstract:

In metaphysics we often wish to talk about extremely general subject matters, like the totality of all objects, be they actual or possible, sets or non-sets. However, standard set theory tells us, for instance, that there cannot be such a thing as the set of absolutely everything. Some have tried to overcome this prohibition by making modifications to logic or to standard set theory, or by renouncing absolutely general discourse altogether. Here, we explore an alternative route: to employ Quine’s ‘New Foundations’ system of set theory, which seems to be particularly well suited to deal with large sets such as are interesting for the metaphysician. We show how NF can be invoked to answer two difficulties about high-level generality: the problem of the domain for absolutely general quantification and the problem of the totality of actual and possible non-sets. We also address, in passing, a common objection about the intuitive motivation of NF.

KEYWORDS: New Foundations; absolute generality; set theory; modality

According to the so-called ‘All-in-One Principle’, the objects of any domain of discourse make up a set or at least some set-like object (Rayo & Uzquiano, 2016: 6). This thesis underpins the possibility of giving a standard Tarskian semantics for large fragments of our language. There is, then, a certain presumption in favor of that principle based on its theoretical utility. However, we start running into trouble once we consider the kind of domains the metaphysician is interested in, like Reality as a whole. Consider absolutely general quantification, i.e., quantification over absolutely everything. This is *prima facie* intelligible; after all, if ontological discourse is to be meaningful, we need to be able to speak not just about these or those things, but about *everything*. The All-in-One Principle entails that the domain of this sort of quantification must be the universal set. However, according to standard Zermelo-Fraenkel set theory (**ZF**), such a set cannot exist.

At least two different strategies are available here if one wants to preserve the All-in-One Principle and continue to affirm the intelligibility of absolutely general quantification. One is to embrace naïve set theory together with an underlying paraconsistent logic (Priest, 1987). A less radical strategy, and the one we advocate, is to employ an alternative (but classical) set theory which admits the universal set. An obvious choice here is Quine’s ‘New Foundations’ (**NF**) system (Quine, 1937), which proves the existence of a set such that everything (including itself) is a member of it. We want to suggest that **NF** is especially well suited to handle domains of discourse appropriate to the level of generality that is common in metaphysics.

However, many have felt that **NF** lacks any intuitive motivation comparable to that which the ‘iterative conception’ of set provides for **ZF** (Boolos, 1983). Linnebo expresses the worry

quite clearly: '[...] all known set theories with a universal set, such as Quine's New Foundations, are not only technically unappealing but have lacked any satisfactory intuitive model or conception of the entities in question' (Linnebo, 2006:156-157).

We believe Linnebo's objection can be answered by distinguishing two pre-theoretical conceptions of set: the *iterative* and the *logical*. **ZF** formalizes the iterative conception: sets understood as the result of an ideal stepwise process of 'collecting' objects; logical sets, on the other hand, are *extensions of concepts*. This distinction is not new, it goes back at least to Gödel (see Wang, 1983). We believe it's the logical conception of sets which underlies **NF**. This can be seen in two ways: (1) **NF** is just a slight modification of naïve set theory, which is widely believed to formalize the conception of sets as extensions of arbitrary concepts or properties (Priest 1987). (2) A distinguishing mark of sets in the 'logical' sense is that they are endowed with a Boolean structure (Shapiro, 1999); and, indeed, the sets of **NF** exhibit such a structure (e. g. in **NF** every set has an absolute complement, something which is impossible in **ZF**). We contend that the logical conception of set serves as an intuitive motivation for **NF** (see Incurvati 2020 for a similar argument), thus answering Linnebo's worry.

Now, although **NF** gives us a universal set, there are other difficulties connected with the kind of discourse that tries to deal with generality. Let's consider one of these problems:

Assuming the All-in-One Principle, we should admit that all objects we find in the domain of a modal discourse (e. g. 'possible objects') form a set. Moreover, in the context of **ZFCU** (**ZFC** + Urelements), the iterative conception of set allows us to have the set of all urelements at the lowest level of the set theoretical hierarchy. Nolan (1996) showed that this assumption together with a plausible principle of recombination yields a paradox in the context of Lewis' actualism. The principle in question says this:

(R) 'For any objects in any worlds, there exists a world that contains any number of *distinct* duplicates of all of those objects' (Nolan, 1996: 242).

In the context of Lewis' actualism, **(R)** entails the following claim (Menzel, 2014: 1):

(A) For any cardinal κ , there are at least κ urelements.

More recently, Sider (2009) has shown that, in the context of Williamson's necessitism, **(R)** entails this other claim:

(B) For any cardinal κ , if it is possible that at least κ urelements exists, then there exists at least κ possible urelements.

It's now easy to see that both **(A)** and **(B)** lead to contradiction together with the assumption of the All-in-One Principle, since in **ZFCU** it is not possible to have a set larger than every cardinal^[1].

In order to avoid this result while retaining the iterative conception of sets, Menzel (2014) has proposed to modify **ZFCU**. According to him, the set of absolutely all urelements is not problematic for the same reasons as the universal set is: the problem with the latter is that it's 'too high up' in the hierarchy, whereas the problem with the former is that it's too 'wide' (Menzel, 2014: 10). Consequently, he proposes to restrict the Replacement Schema for it to apply only to 'small' sets, since that restriction precludes the possibility of deriving a contradiction from the conjunction of the All-in-One Principle and claims such **(A)** or **(B)**.

This strategy could be attractive if one wishes to maintain the iterative conception of set as the only viable one. But here, again, appealing to **NF** turns out to be a better option, for **NFU** (**NF** + Urelements) does not only allow us to have the universal set, but also a set of urelements of arbitrarily large size^[2]. So, the metaphysician who wishes to countenance a set of all urelements (actual and possible) may help herself to **NF** and do so without fear of paradox.

In summary, we have dealt with two problems related to high levels of generality in metaphysical discourse which are notorious for escaping the grip of standard set theory: the problem of the universal set and the problem of the set of all urelements. We have suggested that turning to **NF** may be a fruitful way of admitting these large sets into metaphysical discourse.

[1] In Lewis' actualism, modality is reduced to extensional facts about the pluriverse, so the proof of the inconsistency of the All-in-One Principle together with **(A)** is relatively simple. However, we should make some modifications in the case of **(B)**, since modality in necessitism cannot be reduced to extensional facts (see Sider, 2009).

[2] We thank Thomas Forster for pointing this out to us in personal correspondence.

REFERENCES:

- Boolos, G. (1983). 'The iterative concept of set'. In Putnam, H. & Benacerraf, P. (Eds.), *Philosophy of Mathematics*, 486 – 502. Cambridge: Cambridge University Press.
- Forster, T. (2018). 'Quine's New Foundations'. In Zalta, E. N. (Ed.), *The Stanford Encyclopedia of Philosophy* (Summer 2019 Edition). URL: <https://plato.stanford.edu/entries/quine-nf/>

- Incurvati, L. (2020). *Conceptions of Sets and the Foundations of Mathematics*. Cambridge: Cambridge University Press.
- Linnebo, Ø. (2006). ‘Sets, Properties and Unrestricted Quantification’. In Rayo, A. & Uzquiano, G. (Eds.), *Absolute Generality*, 149 – 178. Oxford: Oxford University Press.
- Menzel, C. (2014). ‘Wide Sets, ZFCU and the Iterative Conception’, *Journal of Philosophy*, 111 (2): 57 – 83. doi: 10.5840/jphil201411124
- Priest, G. (1987). *In Contradiction: A Study of the Transconsistent*. Oxford: Oxford University Press.
- Quine, W.V.O. (1937). “New Foundations for Mathematical Logic”, *The American Mathematical Monthly*, 44 (2): 70 – 80. doi: 10.1080/00029890.1937.11987928.
- Rayo, A. & Uzquiano, G. (Eds.) (2006). *Absolute Generality*. Oxford: Oxford University Press.
- Shapiro, S. (1999). ‘Do not claim too much: Second-order logic and first-order logic’, *Philosophia Mathematica*, 3 (7): 42 – 64. doi: 10.1093/phimat/7.1.42
- Sider, T. (2009). ‘Williamson’s Many Necessary Existents’, *Analysis*, 69 (2): 250 – 258. doi: 10.1093/analys/anp010
- Wang, H. (1983). ‘The concept of set’. In Putnam, H. & Benacerraf, P. (Eds.), *Philosophy of Mathematics*, 530 – 570. Cambridge: Cambridge University Press.
- Williamson, T. (2003). ‘Everything’, *Philosophical Perspectives*, 17 (1): 415 – 465. doi: 10.1111/j.1520-8583.2003.00017.x

[1] For the difference between ‘possible object’ and ‘merely possible object’, see Williamson, 2013.

[2] In Lewis’ actualism, modality is reduced to extensional facts about the pluriverse, so the proof of the inconsistency of the All-in-One Principle together with **(A)** is relatively simple. However, we should make some modifications in the case of **(B)**, since modality in necessitism cannot be reduced to extensional facts (see Sider, 2009).

An Ouroboros of Team Temporal Logics

Max Sandström, max.sandstrom@helsinki.fi

University of Helsinki, Department of Mathematics and Statistics, Helsinki, Finland

This talk is based on joint work with Juha Kontinen.

The aim of this talk is to present a squeezing result in the complexity of logics capturing hyperproperties, which is a consequence of our work and previous results by Lück. The titular ouroboros, the snake that eats its own tail, arises from a chain of translations from TeamLTL^s to third-order arithmetic Δ_0^3 via second order logic SO and back to TeamLTL^s . This entails that not only does TeamLTL^s have the same complexity for the model checking problem as Δ_0^3 , but the same holds for all logics expressively between TeamLTL^s and SO.

Linear temporal logic (LTL), as the name suggests, is a logic for capturing linear models of time. Despite being a simple logic, it was proven useful in theoretical computer science, when Amir Pnueli in 1977 connected it to system verification through the model checking problem, and the role of the logic and that problem in formal verification has been studied extensively since then [5]. In the most prevalent application the logic is used to check whether a system fulfils some given specifications. However, the logic cannot express the totality of the relevant specifications a system may have, since it cannot express dependencies between the computation traces. Among such hyperproperties, as this kind of properties were named by Clarkson and Schneider in 2010, are for instance noninterference and secure information flow, as well as other properties important in cybersecurity [1]. This context has been the motivation for the recent surge of interest in extensions of LTL.

One approach to extending LTL to capture hyperproperties uses team semantics. Team semantics is a framework for extending logics by considering truth through regarding teams of assignments, instead of single assignments, as the defining feature for the satisfaction of a formula [6, 2]. When applied to LTL, this framework provides an approach to capture hyperproperties. Krebs et al. in 2018 introduced two semantics for LTL under team semantics: the synchronous semantics and the asynchronous variant that differ on the interpretation of the temporal operators [3]. In team semantics the temporal operators advance time on all traces of the current team and with the disjunction \vee , a team can be split into two parts during the evaluation of a formula, hence the nickname splitjunction.

Recently it was shown by Lück that the complexity of satisfiability and model checking of synchronous TeamLTL with Boolean negation \sim is equivalent to the decision problem of third-order arithmetic [4] and hence highly undecidable.

In for the translation to first-order team semantics, we need a first-order model. To that end, let $T = \{t_j \mid j \in J\}$ be a set of traces and $f: T \rightarrow \mathbb{N}$ a constant function. In order to simulate TeamLTL^s and its extensions in first-order team semantics we encode T under f by a

first-order structure $\mathcal{M}_{T,f}$ of vocabulary $\{\leq\} \cup \{P_i \mid p_i \in \text{AP}\}$ such that

$$\begin{aligned} \text{Dom}(\mathcal{M}_{T,f}) &= T \upharpoonright f \times \mathbb{N} \\ \leq^{\mathcal{M}_{T,f}} &= \{(t_i, n), (t_j, m) \mid i = j \text{ and } n \leq m\} \\ P_i^{\mathcal{M}_{T,f}} &= \{(t_k, j) \mid p_i \in t_k(j)\}. \end{aligned}$$

We can then define a translation ST_x^* , for which we can show the following.

Theorem 1 *Let φ be a TeamLTL^s(\sim)-formula. Then $T \models \varphi \Leftrightarrow \mathcal{M}_{T,f} \models_{S_{T,f}^x} ST_x^*(\varphi)$ for all non-empty T and increasing functions f .*

Similarly we can define a translation from SO to Δ_0^3 .

Theorem 2 *Let $\phi \in \text{SO}$ be a sentence. Then there exists a formula $\text{Tr}(\phi)(\mathbf{a})$ of Δ_0^3 such that for all trace sets T :*

$$\mathcal{M}_{T,f} \models \phi \iff (\mathbb{N}, +, \times, \leq, 0, 1) \models \text{Tr}(\phi)(\mathcal{A}_T/\mathbf{a}),$$

where $\mathcal{M}_{T,f}$ is defined as previously.

Now due to the equivalence result by Lück, we force the complexity of all effectively intermediary logics.

Theorem 3 *The model checking and satisfiability problems of any logic L effectively residing between TeamLTL^s(\sim) $\leq L \leq \text{SO}$ are equivalent to Δ_0^3 under logspace-reductions.*

References

- [1] Clarkson, M.R., Schneider, F.B.: Hyperproperties. *Journal of Computer Security* **18**(6), 1157–1210 (2010)
- [2] Krebs, A., Meier, A., Virtema, J.: A team based variant of CTL. In: *TIME 2015*. pp. 140–149 (2015). <https://doi.org/10.1109/TIME.2015.11>, <http://dx.doi.org/10.1109/TIME.2015.11>
- [3] Krebs, A., Meier, A., Virtema, J., Zimmermann, M.: Team Semantics for the Specification and Verification of Hyperproperties. In: Potapov, I., Spirakis, P., Worrell, J. (eds.) *MFCS 2018*. vol. 117, pp. 10:1–10:16. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany (2018)
- [4] Lück, M.: On the complexity of linear temporal logic with team semantics. *Theoretical Computer Science* (2020)
- [5] Pnueli, A.: The temporal logic of programs. In: *18th Annual Symposium on Foundations of Computer Science*. pp. 46–57. IEEE Computer Society (1977)
- [6] Väänänen, J.: *Dependence Logic*. Cambridge University Press (2007)
- [7] Virtema, J., Hofmann, J., Finkbeiner, B., Kontinen, J., Yang, F.: Linear-time temporal logic with team semantics: Expressivity and complexity. *CoRR* **abs/2010.03311** (2020)

Modal team logics for modelling free choice inference

Aleksi Anttila, Maria Aloni, and Fan Yang

Free choice is a natural language phenomenon whereby disjunctive sentences appear to license conjunctive inferences:

You may have coffee or tea.

is often interpreted as implying

You may have coffee and you may have tea.

Aloni [1] proposes a Bilateral State-based Modal Logic (BSML) to account for free choice. BSML makes use of team semantics: formulas are interpreted with respect to sets of possible worlds (*states/teams*) rather than the single worlds employed in standard Kripke semantics.

BSML extends classical modal logic with a non-emptiness atom NE originally introduced in the context of propositional team logics [3]. NE is true in a team just in case the team is non-empty. Since contradictions are only true in the empty team, NE allows for the representation of a *pragmatic enrichment* of formulas by the pragmatic principle “avoid stating a contradiction”. Free choice inferences are derived as entailments involving pragmatically enriched formulas.

This talk expands upon [2]. We present complete natural deduction axiomatizations for BSML and two extensions: BSMLI, an extension with the inquisitive disjunction, and BSMLE, an extension with an emptiness operator \emptyset ; $\emptyset\phi$ is true in a team just in case ϕ is true in the team or the team is empty. These axiomatizations are modal extensions of systems for propositional team logics developed in [3]; the key new contribution of the BSML systems is the provision of rules governing the interaction of NE and the modal operators. For the completeness proofs we adapt a standard team logic completeness proof strategy involving characteristic formulas for teams to the modal setting.

The characteristic formulas are also used for proving expressive power results for the logics. BSMLI is expressively complete for the class of all team properties invariant under a type of bisimulation for teams. BSML is union-closed but not expressively complete for the class of all union-closed team properties invariant under the type of bisimulation considered. This motivates our introduction of \emptyset : BSMLE is expressively complete with respect to this natural class of properties.

References

- [1] Maria Aloni. *Logic and Conversation: the Case of Free Choice*. Manuscript, University of Amsterdam, 2022. URL: <https://www.marialoni.org/resources/Aloni2022.pdf>.
- [2] Aleksi Anttila. “The Logic of Free Choice. Axiomatizations of State-based Modal Logics”. MSc thesis. University of Amsterdam, 2021.
- [3] Fan Yang and Jouko Väänänen. “Propositional team logics”. In: *Annals of Pure and Applied Logic* 168.7 (July 2017), pp. 1406–1441.

Demystifying Attestation in Intel Trust Domain Extensions (TDX) via Formal Verification

Muhammad Usama Sardar and Christof Fetzer

Chair of Systems Engineering, Institute of Systems Architecture
Technische Universität Dresden, Dresden, Germany
Correspondence email: muhammad_usama.sardar@tu-dresden.de

1 Abstract

The proposed talk promises to introduce an emerging and exciting domain of *confidential computing* with our recent work on the formalization of one of the most critical processes, namely *remote attestation*, and stimulate discussions on promising directions of future research and potential collaborations.

2 Introduction

Intel Trust Domain Extensions (TDX) is the next-generation confidential computing offering of Intel. One of the most critical processes of Intel TDX is the remote attestation mechanism. Since remote attestation bootstraps trust in remote applications, any vulnerability in the attestation mechanism can therefore impact the security of an application. Hence, we investigate the use of formal methods to ensure the correctness of the attestation mechanisms. The symbolic security analysis of remote attestation protocol in Intel TDX reveals a number of subtle inconsistencies found in the specification of Intel TDX that could potentially lead to design and implementation errors as well as attacks. These inconsistencies have been reported to Intel and Intel is in process of updating the specifications. We also explain how formal specification and verification using ProVerif could help avoid these flaws and attacks.

3 Outline

The proposed talk will specifically address the following questions:

- What is confidential computing? How does it compare with the existing state-of-the-art technologies, such as Homomorphic Encryption?
- Why remote attestation is critical in confidential computing?
- What were the challenges in the formal specification of remote attestation in Intel Trust Domain Extensions (TDX)?
- How we drive formal methods to practice for the automated verification of security properties of remote attestation protocols in Intel TDX?

- What are interesting open challenges of relevance for logic community for formal verification of remote attestation in confidential computing?

Proof theoretic relations between different versions of Higman's and Kruskal's theorem

Gabriele Buriola¹ and Peter Schuster¹

¹*Department of Computer Science, University of Verona, Italy*

Higman's lemma and Kruskal's theorem are two of the most celebrated results in the theory of well-quasi orders. In his seminal paper [5] Higman obtained what is known as Higman's lemma as a corollary of a more general theorem, dubbed here Higman's theorem. While the lemma deals with finite strings over a well-quasi order (and so, implicitly, with only the binary operation of juxtaposition), the theorem is about abstract operations of arbitrary high arity, covering a far more extensive spectrum of situations.

Kruskal was well aware of this more general set up; in his time-honoured paper [6] not only did he use Higman's lemma in crucial points of the proof of his own theorem, but also followed the same proof schema as Higman. Moreover, in the very end of his article, Kruskal explicitly stated how Higman's theorem is a special version, restricted to trees of finite degree (i.e., trees with an upper bound regarding the number of immediate successors of each node), of Kruskal's own tree theorem. Although no proof of the reduction is provided, he presented a glossary to properly translate concepts from the tree context of his paper to the algebraic context of Higman's work. The equivalence between restricted versions has subsequently been exposed by Schmidt [9], whereas Pouzet [7] has given, together with that equivalence, an infinite version of Higman's theorem which proves equivalent to the general Kruskal theorem.

Besides its crucial role in theoretical computer science, e.g., for term rewriting [1,2], Kruskal's theorem characterizes some relevant parts of proof theory. In the vein of the Friedman–Simpson programme of reverse mathematics [10], considerable efforts have been undertaken in order to properly understand the proof-theoretic strength of the tree theorem, as well as of some of its most important versions [3,4,8]; nevertheless, quite a few aspects remain obscure.

We revisit the aforementioned equivalences to obtain a clear view of the proof-theoretical relations between the different versions of Higman's and Kruskal's theorem, paying particular attention to the former's algebraic formulation. The ultimate goal is to complete the picture within the frame of reverse mathematics and ordinal analysis, following [8].

Keyword: Higman's Lemma, Kruskal's Theorem, Proof Theory, Reverse Mathematics.

References

- [1] N. Dershowitz. Orderings for term-rewriting systems. *Theoretical Computer Science*, 17(3):279–301, 1982.
- [2] N. Dershowitz. Termination of rewriting. *Journal of Symbolic Computation*, 3(1):69–115, 1987.
- [3] A. Freund. From Kruskal’s theorem to Friedman’s gap condition. *Mathematical Structures in Computer Science*, 30:952–975, 2020.
- [4] J.H. Gallier. What’s so special about Kruskal’s theorem and the ordinal Γ_0 ? A survey of some results in proof theory. *Annals of Pure and Applied Logic*, 53(3):199–260, 1991.
- [5] G. Higman. Ordering by divisibility in abstract algebras. *Proc. London Math. Soc.* 3, 2:326–336, 1952.
- [6] J.B. Kruskal. Well-quasi-ordering, the tree theorem, and Vazsonyi’s conjecture. *Trans. Amer. Math. Soc.*, 95:210–225, 1960.
- [7] M. Pouzet. Applications of well quasi-ordering and better quasi-ordering. In *Graphs and order*, pages 503–519. Springer, 1985.
- [8] A. Rathjen and A. Weiermann. Proof-theoretic investigations on Kruskal’s theorem. *Annals of Pure and Applied Logic*, 60:49–88, 1993.
- [9] D. Schmidt. Well-orderings and their maximal order types. 1979. Habilitationsschrift, Mathematisches Institut der Universität Heidelberg. Also in P. Schuster, M. Seisenberger and A. Weiermann, editors, *Well-Quasi Orders in Computation, Logic, Language and Reasoning*, pages 351–391, 2020.
- [10] S. G. Simpson. *Subsystems of Second Order Arithmetic*. Perspectives in Logic. Cambridge University Press, 2 edition, 2009.

Geometric Models and Cone Semantics: An Overview

Victor Lacerda, Ricardo Guimarães, and Ana Ozaki

University of Bergen

1 Introduction

Knowledge Graphs (KGs) have become a popular method for storing semantic data using triples (subject-predicate-object) called facts. The facts are written as text files in a machine readable format where automated reasoning is possible: that is, to construct new knowledge via deductions from previously known facts.

Although popular public KGs such as Wikidata¹ contain millions of facts, they are far from complete. Manually adding new facts to KGs is a demanding and time consuming process. This has sparked interest in finding ways of employing Knowledge Graph embedding techniques to represent logical statements into vector spaces. Under this representation they can be used in conjunction with ML approaches in order to perform tasks such as automated knowledge graph completion, relation extraction, and others [Gutiérrez-Basulto and Schockaert, 2018, Özgep et al., 2020].

We give an overview of two approaches for representing logical formulae in vector spaces, highlighting their strengths and weaknesses, with directions for future research in the topic, emphasizing the need for dealing with uncertainty.

2 Geometric Models

Geometric Models form a novel framework for dealing with embeddings with a geometric interpretation in which relations are represented as convex regions in a space of tuples. It has been put forward as a way of expressing more complex rules when compared to classical models [Gutiérrez-Basulto and Schockaert, 2018]. The embedding is achieved through the following interpretation:

Definition 1. Let \mathcal{R} be a set of relation names and $\mathbf{X} \subseteq \mathbf{C} \cup \mathbf{N}$ be a set of objects. An m -dimensional geometric interpretation η of $(\mathcal{R}, \mathbf{X})$ assigns to each k -ary relation name R from \mathcal{R} a region $\eta(R) \subseteq \mathbb{R}^{k \cdot m}$ and to each object o from \mathbf{X} a vector $\eta(o) \in \mathbb{R}^m$.

Let $\mathcal{A} = \{Dog(fido), Mammal(fido)\}$ be a set of facts and $\mathcal{T} = \{Dog(X) \rightarrow Mammal(X)\}$ a set of rules. Figure 1 is, then, a geometric model of the fact that if something is a dog, then it is a mammal. Since *Dog* and *Mammal* are unary predicates, they are represented as (overlapping) intervals and not as convex polygons in \mathbb{R}^2 as it would be the case for a binary predicate. The individual *fido* is contained in the real interval between the two predicates, indicating that it is both a mammal and a dog, which satisfies our rule.

¹<https://wikidata.org/>

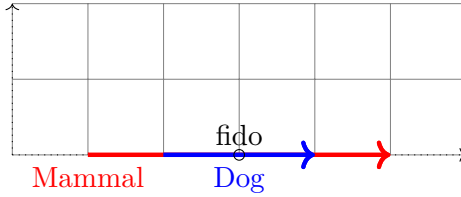


Figure 1: A geometric model

3 Cone Semantics

Cone Semantics have been put forward as a way of embedding ontologies while providing restricted forms of existential and universal quantifiers as well as concept negation and concept disjunction [Özçep et al., 2020].

Definition 2. A Boolean al-cone interpretation \mathcal{I} is a structure $(\Delta, (\cdot)^I)$ where Δ is \mathbb{R}^n for some $n \in \mathbb{N}$, and where $(\cdot)^I$ maps each concept symbol C to some al-cone and each individual a to some element in $\Delta \setminus \{\vec{0}\}$. The interpretation of $\perp = \{\vec{0}\}$. The notions of an al-cone being a model and that of entailment are defined as in the classical case above, but using al-cone interpretations.

Under this method, a geometric interpretation is achieved through axis-aligned cones (al-cones). All al-cones are convex, and are preserved under intersection, polarity, and other operations. The mapping of concepts into al-cones and the use of polarity to define negation allows us to obtain partial models. We are able to represent the fact that we do not know whether a certain individual belongs to a concept or not, a desirable feature when dealing with noisy data.

4 Discussion

We compare and contrast the different strengths of weaknesses of the above described models. Despite the encouraging advance represented by the Geometric Model framework, using convex regions leads to issues with expressing the negation of concepts. This is an essential feature of any sufficiently expressive knowledge base, including those that utilize the DL \mathcal{ALC} .

Cone Semantics does not incur into the issue of being unable to represent negation. Although the results are promising, there is currently no way of representing roles using axis aligned cones, which severely restricts the types of rules that might be expressed in the TBox of an ontology. Regardless of this limitation. As such, new insights are required to overcome the weaknesses in these approaches.

References

- Víctor Gutiérrez-Basulto and Steven Schockaert. From knowledge graph embedding to ontology embedding? an analysis of the compatibility between vector space representations and rules. In *Proceedings of KR*, 2018.
- Özgür Lütfü Özçep, Mena Leemhuis, and Diedrich Wolter. Cone semantics for logics with negation. In Christian Bessiere, editor, *IJCAI*, pages 1820–1826. ijcai.org, 2020.

Product of effect algebras with \sqrt{I} and their Logic

Konstantin Shishov

Quantum logic originally started as an algebraic structure of a set of projective operators in Hilbert space. These structures were later called orthomodular lattices, which were actively investigated. However, most of the attempts to build any adequate logic on the basis of this lattice for various reasons were unsuccessful.

The effect algebras appear in the 90s thanks to D.Foulis and M.Bennett [2]. The effect algebras is to be a generalisation of many important for quantum physics and quantum logics algebraic structures, including orthomodular lattices.

Investigations in the field of residual effect algebras appeared to be the most attractive area for modern quantum logicians. This research would will make possible to connect quantum logic with other substructural logics.

Logic of Lattice Effect algebras was introduced in [1] and [4] as logic of residuated lattice effect algebras. The main feature of these logics is the presence of a logical connective of implication, which is unambiguously can be expressed as Sasaki arrow (or Sasaki projection) with good properties. The introduction of an implication with deductive properties and good axiomatisability of Lattice Effect algebras Logic became possible only thanks to a well-constructed residual algebra of effects.

In this work we expand the residuated effect algebra and their logic by the unary operation \sqrt{I} (*square root of the inverse*). As noted in [3]: the square root of the inverse can be seen as a kind of "tentative inversion": by applying it twice to a given element a , we obtain the inverse a' of the element itself. In the standard algebra, for example, we have:

1. $\sqrt{I}\langle a, b \rangle = \langle b, 1 - a \rangle$;
2. $\sqrt{I}\sqrt{I}\langle a, b \rangle = \langle 1 - a, 1 - b \rangle = \langle a, b \rangle'$;

In order to add this operation to our logic, we consider the algebra of this logic, extended by the square root of the inverse. For this, we consider the product of effect algebras defined as follows:

PE is the algebra $([0, 1] \times [0, 1], \oplus^E, \lrcorner^E, 0^E, 1^E)$, where:

1. $(a_1, b_1) \oplus^E (a_2, b_2) = (\min(1, a_1 + a_2), \frac{1}{2})$;
2. $(a_1, b_1) \lrcorner^E = (1 - a_1, 1 - b_1)$;
3. $0^E = \langle 0, \frac{1}{2} \rangle$;
4. $1^E = \langle 1, \frac{1}{2} \rangle$.

\leq is defined in a standard pairwise way:

$(a_1, b_1) \leq^E (a_2, b_2)$ iff $a_1 \leq a_2$ and $b_1 \leq b_2$

We expand PE by implication:

$$\langle a_1, b_1 \rangle \rightarrow^E \langle a_2, b_2 \rangle := \langle a_2, b_2 \rangle + (\langle a_1, b_1 \rangle \vee \langle a_2, b_2 \rangle)',$$

and by square root of the inverse in the same sense as it was defined above.

Our report will present the main properties of the resulting algebra and logical system, which are obtained at the current stage of work.

References

- [1] Chajda I., Halas R., Langer H. The logic induced by effect algebras. *Soft Comput* 24, pp. 14275 - 14286 (2020)
- [2] Foulis D.J., Bennett M.K. Effect algebra and unsharp quantum logic. *Foundations of Physics*, **24(10)**, 1994, pp. 1331-1352.
- [3] Paoli F., Ledda A., Giuntini R., Freytes H. On Some Properties of quasi-MV Algebras and \sqrt{I} quasi-MV Algebras. *Reports on Mathematical Logic*, 2013, - RML. 10.4467/20842589RM.13.001.1201.
- [4] Rafee Rad S., Sharafi A.H., Smets S.A. Complete Axiomatisation for the Logic of Lattice Effect Algebra // *Int J.Theor.Phys.*(2019)

A Librationist Set Theory Extending Classical Set Theory

Frode Alfson Bjørdal

We present revisions to the librationist system \mathcal{L} (“pound”), cfr. (Bjørdal, 2012 & 2016), so that the revised system \mathcal{B} (“libra”) extends classical set theory.

A formal language, with signature $\circ, c_0, \dots, c_{n-1}$, and the universal quantifier and the Sheffer stroke as primitive operators, is employed. \circ is the set forming variable-cum-formula operating operator, and c_1, \dots, c_n are primitive constants; primitive variables v' are defined by the clause: v is a variable and a variable followed by \bullet is a variable. $a \in b \stackrel{df}{=} ba$ and $\{v|A\} \stackrel{df}{=} \circ vA$.

We presuppose a number theoretic point of view according to which \bullet denotes one, v denotes two, \downarrow denotes three, \forall denotes four, \circ denotes five and, finally, that c_i , for $1 \leq i \leq n$, denotes the natural number five plus i . Once n is fixed, a bijective base- k numeration, where k equals five plus i , is used for the primitive number denoting symbols. The positive symbols smaller than $k+1$, just presented, are primitive symbols, and synonymous with the shortest positive numerals in the $(k+1)$ -*simal* numeral system. The apposition of two bijective base- k numerals m and n , is another bijective base- k numeral $m \hat{\ } n$, defined by associative concatenation with value $m \cdot k^{\ell(n)} + n$, where $\ell(n) = \lfloor (\log_k(n+1)(k-1)) \rfloor$ expresses the length of the numeral needed for number n in the bijective base- k numeral system.

The existential quantifier and the other connectives are defined in a normal way, and some standard formation rules presupposed. In the presentation, variable signs beyond the austere ones are used, and boldface letters, as \mathbf{a} , represent terms which may have no free variables. All strings of symbols, and so also formulas, denote one unique natural number, as no symbol denotes zero. The number theoretic approach here, improves upon (Bjørdal, 2016).

Identity is taken in the Leibnizian-Russellian sense, so that:

1 Definition. $\mathbf{a} = \mathbf{b} \stackrel{df}{=} \forall x(\mathbf{a} \in x \rightarrow \mathbf{b} \in x)$

As with (Gandy, 1959), and in much literature on *non-classical* set theories, and so-called *property-theories*, as (Gilmore, 1974) and (Cantini, 1996) and others, abstracts are used, for the principle of extensionality does not hold. All conditions define a set, so for any formula A , it is an axiom that $\exists x(x = \{x|A\})$.

Stipulate:

2 Definition. For term \mathbf{a} , let $A^{\mathbf{a}}$ signify that all variables bound in formula A , are bound to \mathbf{a} , and let $\{x|A\}^{\mathbf{a}}$ be $\{x|x \in \mathbf{a} \wedge A^{\mathbf{a}}\}$.

Assume:

3 Definition. Condition $\alpha(x, y)$ is extent-functional $\stackrel{df}{=} (\alpha(x, y) \wedge \alpha(x, z) \rightarrow \forall w(w \in y \leftrightarrow w \in z))$

We suppose that \mathbf{b} contains \mathbf{a} just if $\mathbf{a} \in \mathbf{b}$, that \mathbf{G} is one of the privileged primitive constants in the signature, and state axiomatic principles to fill \mathbf{G} :

(1) \mathbf{G} contains $\{x|x \in x \wedge x \notin x\}^{\mathbf{G}} = \{x|x \in \mathbf{G} \wedge (x \in x \wedge x \notin x)\}$

- (2) \mathbf{G} contains $\{\mathbf{a}, \mathbf{b}\}^{\mathbf{G}} = \{x \mid x \in \mathbf{G} \wedge (x = \mathbf{a}^{\mathbf{G}} \vee x = \mathbf{b}^{\mathbf{G}})\}$ if it contains $\mathbf{a}^{\mathbf{G}}$ and $\mathbf{b}^{\mathbf{G}}$
- (3) $\exists u[\forall x(\neg\exists y(y \in x) \rightarrow x \in u) \wedge \forall x\forall y(x \in u \wedge \forall z(z \in y \leftrightarrow \forall w(w \in z \rightarrow w \in x)) \rightarrow y \in u)]^{\mathbf{G}}$
- (4) if \mathbf{G} contains $\mathbf{a}^{\mathbf{G}}$, it contains $\{x \mid \exists y(x \in y \wedge y \in \mathbf{a})\}^{\mathbf{G}}$,
- (5) \mathbf{G} contains $\{x \mid x \subset \mathbf{a}\}^{\mathbf{G}}$ if it contains $\mathbf{a}^{\mathbf{G}}$,
- (6) $\mathbf{a} \in \mathbf{G}$ only if $(\exists y(y \in \mathbf{a}) \rightarrow \exists y(y \in \mathbf{a} \wedge \neg\exists z(z \in \mathbf{a} \wedge z \in y)))^{\mathbf{G}}$ (Foundation);
- (7) if \mathbf{G} contains $\mathbf{a}^{\mathbf{G}}$, and $\alpha(x, y)^{\mathbf{G}}$ is an extent-functional first order condition upon x and y , then \mathbf{G} contains $\{y \mid \exists x(x \in \mathbf{a} \wedge \alpha(x, y))\}^{\mathbf{G}}$. (Replacement).

Version (3) of infinity is an adaptation from (Scott, 1961), page 117, to conform with his construction to support the conclusion drawn, in the antepenultimate paragraph, as it depends upon the interpretative power of the system we presuppose here.

The semantics is by a set theoretic adaptation of a Gupta-Herzberger style process upon ordinals, taken for example as in (Bjrdal 2012). Let Ξ be a function from ordinal numbers to real numbers, i.e. sets of natural numbers, which given our numerical point of view are maximal consistent sets of formulas in the language of \mathfrak{W} , such that for all ordinal numbers γ , $\Xi(\gamma) \Vdash (1) \wedge (2) \wedge (3) \wedge (4) \wedge (5) \wedge (6) \wedge (7)$; and $\Xi(\gamma) \Vdash \downarrow AB$ just if neither $\Xi(\gamma) \Vdash A$ nor $\Xi(\gamma) \Vdash B$; and $\Xi(\gamma) \Vdash \forall x A(x)$ just if $\Xi(\gamma) \Vdash A(\mathbf{a}/x)$ for all \mathbf{a} substitutable for x in A ; and for arbitrary $\beta > 0$, $\Xi(\beta) \Vdash u \in \{u \mid A\}$ just if $\Sigma\gamma < \beta \Pi\delta(\gamma \leq \delta < \beta \Rightarrow \Xi(\delta) \Vdash A)$. Also, for all ordinals γ , $\Xi(\gamma) \Vdash \forall x(x \in \{x \mid x \in \mathbf{G}\} \rightarrow x \in \mathbf{G})$; the latter condition amounts to the assumption that \mathbf{G} is not *paradoxical*, in the sense of the paragraph after the next.

Let us call such a function as Ξ an *attractive* function. A formula A of \mathfrak{W} is valid, $\models A$, just if it for all attractive functions Ξ' at the *closure ordinal* Υ , of the Herzberger-Gupta style revisionary process, holds that $\Xi'(\Upsilon) \Vdash u \in \{u \mid A\}$, or it for all attractive functions Ξ' at the closure ordinal Υ holds that $\Xi'(\Upsilon) \Vdash u \notin \{u \mid \neg A\} \wedge u \notin \{u \mid A\}$.

A formula A of \mathfrak{W} is *paradoxical* just if $\models A$ and $\models \neg A$, and \mathbf{a} is paradoxical just if the formula $\mathbf{b} \in \mathbf{a}$ is paradoxical for some \mathbf{b} .

\mathfrak{W} accounts, via \mathbf{G} , for a version ZF^- of Zermelo-Fraenkel set theory, ZF , minus extensionality, and some more on this is related in the antepenultimate paragraph.

But there are many sets beyond \mathbf{G} in \mathfrak{W} , as e.g. distinct non-paradoxical universal sets, and $R = \{x \mid x \notin x\}$. As in (Bjrdal, 2012 & 2016), paradoxicalities are resolved in an *ultra-consistent*, or *para-coherent*, manner, as both $R \in R$ and $R \notin R$ are valid in the semantics just stated for \mathfrak{W} . Unlike *paraconsistent* approaches, which only justify fragments of logic and mathematics, \mathfrak{W} *extends* classical logic, and does so *sedately* in the sense that (i) if A is valid in \mathfrak{W} , then $\neg A$ is not valid in classical logic and (ii) if A is valid in classical logic then A is valid in \mathfrak{W} .

\mathfrak{W} avoids the counter intuitive categoricity property that for any formula A , either A is a thesis of \mathfrak{L} or $\neg A$ is a thesis of \mathfrak{L} , resulting from the latter's fixation upon one model, by only allowing an empty beginning to the revisionary Herzberger-Gupta process. In contrast, if e.g. $S = \{x \mid x \in x\}$, neither $S \in S$ nor $S \notin S$ is valid in \mathfrak{W} ; nor is $v = V$ or $v \neq V$.

Let $\omega^{\mathbf{G}}$ be a set in \mathbf{G} which contains, precisely, the finite von Neumann ordinals relativized to \mathbf{G} , and let ϵ be one of the primitive constants. Take ϵ to be semantically justified as a non-paradoxical bijection from $\omega^{\mathbf{G}}$ to the full universe V , like in (Bjrdal, 2012), 349-350.

If \mathbf{a} is extensionally distinct from the universal set, V , then the power set $\mathcal{P}(\mathbf{a}) = \{x \mid x \subset \mathbf{a}\}$ of \mathbf{a} is paradoxical. Cantor's argument for the uncountability of the set of real numbers, equipollent with $\mathcal{P}(\omega^{\mathbf{G}})$, is blocked, as a consequence of the paradoxicality of $\mathcal{P}(\omega^{\mathbf{G}})$.¹ Despite this deviant, though in the light of (Skolem 1922/1961), quite welcome result, we do have the result that the

¹Cfr. (Bjrdal, 2012, 346–351) for a more detailed analysis of the situation.

set of real numbers, when based on $\mathcal{P}(\omega^{\mathbf{G}})$, is not *listable*, as no non-paradoxical function f has precisely $\omega^{\mathbf{G}}$ as domain and the mentioned real numbers as its range. Nevertheless, the non-paradoxical universal set, V , is listable, as there is a non-paradoxical function, from $\omega^{\mathbf{G}}$ onto V , viz. \in .

Notice that $\mathcal{P}(\omega^{\mathbf{G}})^{\mathbf{G}} = \{x|x \in \mathbf{G} \wedge x \subset \omega^{\mathbf{G}}\}$ is *not* paradoxical. \in is a non-paradoxical surjection from $\omega^{\mathbf{G}}$ to $\{x|x \in \mathbf{G} \wedge x \subset \omega^{\mathbf{G}}\}$ which bijects from a subset of $\omega^{\mathbf{G}}$ to $\{x|x \in \mathbf{G} \wedge x \subset \omega^{\mathbf{G}}\}$. But \in is not a function *in* \mathbf{G} . On pain of contradiction, there is no function $f \in \mathbf{G}$ from $\omega^{\mathbf{G}}$ onto $\{x|x \in \mathbf{G} \wedge x \subset \omega^{\mathbf{G}}\}$. Given these circumstances, we may hold that \mathbf{G} “erroneously believes” that $\{x|x \in \mathbf{G} \wedge x \subset \omega^{\mathbf{G}}\}$ is not countable.

This in large part conforms with Skolem’s resolution of the seeming contradiction between the fact that a set theoretic system “says” that there are uncountable sets, and the fact that the system has a countable model. But it does not support Skolem’s conclusion that the notion of *set* is *relative*, for we do not assume that the set \in , which is not contained in \mathbf{G} , is contained in a more comprehensive *classical* set theory.

The theory ZF^- developed above, relative to \mathbf{G} , interprets ZF on account of its divergent version of replacement, appealing to *extent-functionality*, by findings of (Scott, 1961, 130-131). Given (Gödel, 1938 & 1940), the approach suffices to interpret ZFC .

In conclusion, an observation concerning inference modes of \mathfrak{U} :

$$\frac{}{\models^M A} \stackrel{df}{=} \models A \ \& \ \not\models \neg A \qquad \text{maxim, or maximal theorem}$$

$$\frac{}{\models_m A} \stackrel{df}{=} \models A \ \& \ \models \neg A \qquad \text{minor, or minor theorem}$$

The *maxim mode*, which says B is a maxim, if A and $A \rightarrow B$ are maxims, and many other inference modes, can be established; but modus ponens for \models is not a valid mode.

References:

- Björddal, F. A. 2012: Librationist closures of the paradoxes, *Logic and Logical Philosophy*, 21(4): 323-361.
- Björddal, F. A. 2016: Elements of librationism, arXiv:1407.3877
- Cantini, A. 1996: *Logical Frameworks for Truth and Abstraction*, Elsevier.
- Gandy, R. 1959: On the axiom of extensionality II, *Journal of Symbolic Logic*, 24(4): 287-300.
- Gilmore, P. G. 1974: The consistency of partial set theory without extensionality, in J. Selding *et alia*, *To H. B. Curry: Combinatory Logic, Lambda Calculus and Formalism*, Academic Press, New York, 99-123.
- Gödel, K. 1938: The consistency of the axiom of choice and of the generalized continuum-hypothesis, in *Proceedings of the U.S. National Academy of Sciences*, 24: 556-7.
- Gödel, K. 1940: *Consistency of the Axiom of Choice and of the Generalized Continuum-Hypothesis with the Axioms of Set Theory*, Princeton University Press.
- Scott, D. 1961: More on the Axiom of Extensionality, in Y. Bar Hillel *et alia*, *Essays on the Foundations of Mathematics*, Hebrew University, Jerusalem: 115-131.
- Skolem, T. 1922: Einige Bemerkungen zu axiomatischen Begründung der Mengenlehre, in *Proceedings of the 5th Scandinavian Mathematical Congress*, Helsinki: 217-232.
- Skolem, T. 1961: Some Remarks on Axiomatized Set Theory, in J. van Heijenoort (ed.) *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*: 290-301; a translation of (Skolem, 1922).